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Martingale Central Limit Theorem and Nonuniformly Hyperbolic Systems

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MARTINGALE CENTRAL LIMIT THEOREM AND NONUNIFORMLY HYPERBOLIC SYSTEMS

A Dissertation Presented

by

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DEDICATION

I would like to dedicate my thesis to my parents and grandmother.

Without your love and support I never would have made it this far.

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ABSTRACT

MARTINGALE CENTRAL LIMIT THEOREM
FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

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In this thesis we study the central limit theorem (CLT) for nonuniformly hyperbolic dynamical systems. We examine cases in which polynomial decay of correlations leads to a CLT with a non-standard scaling factor of $\sqrt{n \ln n}$. We also formulate an explicit expression for the the diffusion constant σ in situations where a return time function on the system is a certain class of supermartingale. We then demonstrate applications by exhibiting the CLT for the return time function in four classes of dynamical billiards, including one previously unproven case, the skewed stadium, as well as for the linked twist map. Finally, we introduce a new class of billiards which we conjecture are ergodic, and we provide numerical evidence to support that claim.

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CHAPTER 1

Introduction

1.1 Statistical properties of dynamical systems

The theory of dynamical systems has its origin in classical and statistical mechanics through the works of Poincaré and Boltzmann. The ultimate goal is to provide a mathematical foundation for statistical mechanics which consists of equilibrium and nonequilibrium systems. Equilibrium systems are usually described by closed Hamiltonian equations. One such class of systems is classical billiard models, in which a point particle moves freely at unit speed until it undergoes elastic collisions with a fixed boundary.

Many mechanical systems with elastic collisions preserving the total momentum and energy of the system reduce to classical billiards. There are many physically motivated variations on billiards. For instance, periodic Lorentz gas describes the motion of electron gases in crystals and serves as a useful model of tagged particle diffusion in a binary mixture [51]. Dispersing billiards [80] were introduced by Ya. Sinai in 1970; his studies were motivated by Boltzmann's Ergodic Hypothesis [12, 13, 79] for hard gases of hard balls. These systems were proven to be hyperbolic and ergodic by Sinai in [80]; in fact, they enjoy exponential decay of correlations and converge to a Brownian motion [26, 38, 91]. However, in applications to physical sciences other issues, known as statistical properties, are of utmost importance.

Let (X, T, μ) be a dynamical system, with X a Riemannian manifold and with the map $T : X \to X$ preserving an invariant probability measure μ. One can study the statistical properties of a real-valued observable function f on X by defining the sequence of random variables $X_i = f \circ T^i$; this sequence is dependent and identically distributed. One intuitive line of inquiry is to study whether statistical properties, such as the central limit theorem, are satisfied for this process. It is often the case that as long as the system in question exhibits sufficiently chaotic behavior, one may expect such limiting theorems to hold.

According to a modern view, chaotic behavior in deterministic systems is caused by the exponential instability of nearby trajectories. The best illustration of this statement is provided by Axiom A diffeomorphisms where the expansion of some directions and the contraction of complementary ones are uniform. In general we will study two-dimensional uniformly or nonuniformly hyperbolic systems. A system is uniformly hyperbolic if every point $x \in X$ has Lyapunov exponents $\chi_1(x) < 0 < \chi_2(x)$, and is nonuniformly hyperbolic if this is true except in a null set of X, where the Lyapunov exponents may be zero. For more information on uniformly hyperbolic systems, see the works of Anasov, Sinai, Bowen, Parry, Pollicott, Viana, and Young in [3, 4, 14, 72, 86, 91]. In fact, properties of uniformly hyperbolic systems are quite well understood.

In certain nonuniformly hyperbolic systems (X, T, μ) it is possible to define a subset $Y \subset X$ which doesn't see the portions of phase space that contribute to the nonuniformity of the hyperbolicity. One can define a return time map on this set,

$$\mathcal{R}(y) = \min\{m \geq 1 : T^m y \in Y\}, \tag{1.1}$$

and an induced map $F : Y \to Y$ which satisfies $F(y) = T^{\mathcal{R}(y)}(y)$. Furthermore, this induces an invariant measure $\mu_Y = \mu/\mu(Y)$. In the examples we study in Chapter 6 the induced dynamical system (Y, F, μ_Y) is uniformly hyperbolic and studying its statistical properties leads to insights about the properties of the original system. In this thesis we will restrict our attention to proving the central limit theorem for the function $\mathcal{R}$. This observable is arguably one of the most physically interesting and relevant, as it provides information about the length of time that trajectories may be stuck in the "bad space" Y, during which nearby trajectories are experiencing little to no instability. This information is important when analyzing the chaotic nature of a system.

Another property of the systems we study is that they exhibit deterministic diffusion. One definition often used in the physical literature states that a system if diffusive if the mean squared displacement grows proportionally to time t, asymptotically as $t \to \infty$. Such systems can often be modeled using the diffusion (or heat) equation, and it is here that the importance of the central limit theorem becomes apparent. The diffusion constant in the partial differential equation is related to the variance of the normal variable in the central limit theorem, and this is one way to see the relationship between the micro- and macro-scale behaviors of such deterministic systems. For an in-depth discussion of deterministic diffusion and billiards see Sanders' thesis [76]. A major advantage of our work is that we are able to give explicit forms for diffusion constants in a variety of nonuniformly hyperbolic systems.

1.2 History and recent developments

One of the key aims of statistical mechanics is to relate the microscopic properties of a fluid to the transport coefficient. These include diffusion coefficients, viscosity, and heat conduction. Recently it has been realized that the statistical properties of deterministic billiards resemble to some extent those of diffusion. Billiards can be regarded as the simplest physical systems in which diffusion, understood as the large-scale transport of mass through the system, can occur, as observed by Bunimovich in [23]. Other transport processes have also been studied in billiards, such as electrical conduction [35, 36], viscosity [23, 29] and heat conduction [1]. Limiting laws in classical hyperbolic systems are better understood and proved or almost proved in quite a few cases. However, only recently have these laws become a main focus of study for nonuniformly hyperbolic systems, so the development of techniques to prove limiting laws is of great mathematical interest.

Ergodic and mixing systems may have quite different statistical properties depending on the rate of mixing. Many mixing systems have slow (polynomial) mixing rates which cause weak statistical properties; this situation commonly arises in nonuniformly hyperbolic systems. The central limit theorem may fail and affect the convergence to a Brownian motion in a proper space-time limit (weak-invariance principle) and many other useful approximations by stochastic processes that play crucial roles in statistical mechanics. Such systems exemplify a delicate transition from regular behavior to chaos. For this reason they have attracted considerable interest in mathematical physics during the past 20 years, see [6, 7, 34, 39, 42, 70, 91, 92] and the references therein.

It is very challenging to estimate the decay rates of correlations for hyperbolic systems with singularities, including chaotic billiards. One main reason is that these systems may have singularities which lead to an unpleasant fragmentation in phase space. More precisely, any unstable manifold may expand locally, but the singularities may cut its images into many pieces; some of them being of much smaller size than the original ones, which requires a long time to recover. Moreover, the differential of the map can also be unbounded, and/or with unbounded distortion, which aggravates the analysis.

Historically, there are mainly three methods to prove exponential decay of correlations as well as other statistical properties for general systems with hyperbolicity.

- **Markov sieves** were designed and used by Bunimovich, Chernov and Sinai [24, 26] to establish the existence of SRB measures, the CLT, and a stretched exponential upper bound on the correlation function for hyperbolic systems with singularities. Young [91, 92] made a breakthrough and proved the existence of SRB measures and the exponential decay of correlations using Markov approximations based on Young's tower construction. Her method was used to prove that dispersing billiards enjoy exponential decay of correlations; see [32, 38, 91].

- **Coupling methods** were first introduced by Doeblin [45] in his work on the convergence to equilibrium on Markov chains. His method was improved by Harris, Bressaud, etc. [18, 53]. In recent papers by Chernov, Dolgopyat [33, 34] and Young [92], the coupling method was established for billiards and successfully used to prove exponential decay of correlations for billiards, as well as other properties. In recent work by Chernov and Zhang [42], the results were extended to general hyperbolic systems with singularities by coupling.

- **Spectral analysis** was an important tool in the study of thermodynamical formalism for the Perron-Frobenius operators by Bowen and Ruelle in early 70's [14, 74]. It was further developed by Baladi, Gouezel, Keller, Liverani, Tsujii, etc., see [6, 7, 30, 31, 60, 62, 84, 85] to study the existence of SRB measures and their statistical properties for various hyperbolic systems, excluding billiards. The unbounded derivatives and the existence of the singularities in Sinai billiards make it very difficult to construct the suitable abstract functional space on which the transfer operator exhibits a spectral gap. In [44], Demers and Zhang made a breakthrough in this direction and solved the open question of proving exponential decay of correlations and various limiting theorems for Sinai billiards using spectral analysis.

Chaotic properties and SRB measures for classical hyperbolic systems with fast decay of correlations have been well understood through the above three parallel tracks, and attention is now shifting to open questions concerning nonuniformly hyperbolic systems with slow decay of correlations. Indeed, the central limit theorem has been proved for a variety of these systems, see for example the works of Bálint, Chernov, Dolgopyat, Gouëzel, Markarian, Szász, and Varjú [8, 9, 41, 83]. The techniques that have been previously utilized are often quite table-specific, requiring geometric calculations based on the particular shape of the table being considered. In this thesis we utilize a martingale difference decomposition technique which can be used to prove the central limit theorem for the return time map $\mathcal{R}$, defined in (1.1), for a wide variety of nonuniformly hyperbolic systems. Additionally, $\mathcal{R}$ will satisfy properties in these systems which allow us to explicitly compute the variance of the limiting normal distribution in the CLT.

1.3 Probability limiting theories

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of random variables defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is a classical result that if the aforementioned sequence is independent and identically distributed with mean $\mathbb{E}(X_1) = \mu$ and variance $\mathrm{Var}(X_1) = \sigma^2$ then the strong law of large numbers and central limit theorem both hold. The strong law of large numbers states that the average of n random variables converges to μ almost surely as n approaches infinity, that is,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mu. \tag{1.2}$$

We denote $S_n := \sum_{i=1}^{n} X_i$ as the partial sum of the stochastic process, then (1.2) implies that for n large the partial sum S_n can be approximated as $S_n \sim n\mu + o(n)$, where $f(n) = o(g(n))$ means (for non-zero $g(n)$) that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.[1] However, when the expected value $\mathbb{E}(X_1) = 0$ the estimation in (1.2) does not give us much information, so one would like to get a better estimation. Moreover, this also relates to the question of the convergence rate of S_n/n to μ. More precisely, one hopes to find a sequence $a_n = o(n)$ as n goes to ∞ such that $S_n/n - \mu = a_n$.

For the case when $\mathrm{Var}(X_1) \in (0, \infty)$, the central limit theorem gives us information about what $\{a_n\}$ looks like. The theorem states that

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} Z, \tag{1.3}$$

where the convergence is in distribution. Here Z is a standard normal variable with standard normal distribution $\mathcal{N}(0, 1)$. This can be rewritten to give an approximation for understanding

[1]We say two sequences $a_n \sim b_n$, if there exist $c_1 < C_2$, such that $c_1 a_n \leq b_n \leq c_2 a_n$.

the partial sum S_n:

$$\frac{S_n}{n} - \mu \sim \frac{\sigma}{\sqrt{n}} Z. \tag{1.4}$$

Thus we have a precise form of the sequence $a_n = \sigma Z/\sqrt{n}$.

However, for the case when $\operatorname{Var} X_n = \infty$, the results due to Marcinkiewicz and Zygmund (Durrett's book Th 2.5.8 and Excercise 2.5.2, [48]) give the following fact.

Lemma 1.1. *Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}(X_1) = 0$ and let $S_n = X_1 + \cdots + X_n$. Then $S_n/n^{1/p} \xrightarrow{a.s.} 0$ if and only if $\mathbb{E}|X_1|^p < \infty$, where $1 < p < 2$.*

Thus there exists $c_n = o(1)$, such that

$$\frac{S_n}{n} - \mu = n^{-p} c_n, \tag{1.5}$$

almost surely.

In particular, an important fact on convergence to the normal distribution is the next result due to Lévy. (Theorem 3.4.6. in Durrett [48, 63]):

Lemma 1.2. *Let $X_1, X_2, \ldots$ be i.i.d. and $S_n = X_1 + \cdots + X_n$. In order that there exist constants a_n and $b_n > 0$ so that $(S_n - a_n)/b_n \to Z$, it is necessary and sufficient that*

$$x^2 \mathbb{P}(|X_1| > x)/\mathbb{E}(|X_1|^2; |X_1| \le x) \to 0 \tag{1.6}$$

For example, assume $\mu = 0$, if $P(|X_1| > x) = x^{-\alpha}$ where $\alpha \in (0, 2)$, then $S_n/n^{\frac{1}{\alpha}}$ converges to a limit which is not a normal variable. However, it is an interesting fact that even when $\operatorname{Var}(X_1) = \infty$, if $P(|X_1| > x) = x^{-2}$, then $S_n/\sqrt{n \ln n}$ converges to a normal variable in distribution.

Much work has been done to extend the above results beyond independent, identically distributed sequences. Analogous theorems exist for sequences which are dependent or are not identically distributed. The same holds true for arrays of random variables which may have these properties, and for martingale sequences or arrays. The assumptions placed on these sequences or arrays influence the variance of the normal random variable in (1.3) and finding bounds or an explicit expression for its variance is an important step in understanding the statistical properties of a given sequence or array or random variables. In particular, many situations we study will exhibit a non-classical scaling of $\sigma\sqrt{n \ln n}$ in (1.3), as opposed to the classical $\sigma\sqrt{n}$.

One major advantage of studying martingales is that while they are not generally independent sequences of random variables, their dependence is "weak enough" that it is possible to generalize results for the i.i.d. case to martingales with certain additional properties. Furthermore, the following result due to Doob [47] gives the study of the statistical properties of martingales even more significance.

Lemma 1.3 (Doob decomposition theorem)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ a filtration of $\mathcal{F}$, and $\{X_n\}_{n\in\mathbb{N}}$ an adapted stochastic process with $\mathbb{E}|X_n| < \infty$ for all $n \in \mathbb{N}$. Then there exists a martingale $\{M_n\}_{n\in\mathbb{N}}$ and an integrable predictable process $\{A_n\}_{n\in\mathbb{N}}$ starting with $A_1 = 0$ such that $X_n = M_n + A_n$ for every $n \in \mathbb{N}$. This decomposition is unique almost surely.*

Note that a process $\{A_n\}_{n\in\mathbb{N}}$ is predictable if A_n is $\mathcal{F}_{n-1}$-measurable for every $n \geq 2$. The above result illustrates the usefulness of studying the statistical properties of martingales. If a stochastic process is adapted to a filtration and each random variable in that process is integrable, then showing the central limit theorem for that stochastic process may reduce to proving an associated martingale central limit theorem, as long as the process $\{A_n\}_{n\in\mathbb{N}}$ is, in a sense, negligibly small. We will see this idea put to use in Chapters 3 and 4.

Many versions of the central limit theorem exist for martingales. We will present two results here that are used often in this thesis and in the existing literature. In [71], Neveu presents the following result.

Lemma 1.4. *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a stationary, ergodic, martingale difference with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. If $\mathbb{E}(Y_1^2) = \sigma^2 < \infty$ and $S_n = Y_1 + \cdots + Y_n$ then*

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow{d} Z \tag{1.7}$$

where Z has the standard normal distribution.

This statement has clear similarities to (1.3), especially upon noting that martingale differences have zero expectation. The sequence is identically distributed as it is stationary, and in addition to being a martingale difference the only further assumption is that the sequence is ergodic, which is equivalent to the strong law of large numbers holding true.

The next result, proved by Hall and Heyde in [52], extends the central limit theorem to martingale difference arrays.

Lemma 1.5. *Let $\{(X_{n,i}, \mathcal{F}_{n,i}), 1 \leq i \leq k_n, n \geq 1\}$ be a zero-mean, square-integrable martingale* *array with differences $Z_{n,i}$, and let σ^2 be an a.s. finite random variable. Suppose that*

$$\max_i |Z_{n,i}| \quad \xrightarrow{p} \quad 0, \tag{1.8}$$

$$\sum_{i=1}^{k_n} Z_{n,i}^2 \quad \xrightarrow{p} \quad \sigma^2, \tag{1.9}$$

$$\mathbb{E}\left(\max_i Z_{n,i}^2\right) \qquad \textit{is bounded in } n, \tag{1.10}$$

and the σ-algebras are nested, that is, $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}$ for $1 \leq i \leq k_n$, $n \geq 1$.
Then $S_{n,k_n} = \sum_i Z_{n,i} \xrightarrow{d} Z$, where the random variable Z has characteristic function $\mathbb{E}\left(e^{-\sigma^2 t^2/2}\right)$.

If σ^2 is a constant, the last statement is equivalent to saying Z has distribution $\mathcal{N}(0, \sigma^2)$. Hall and Heyde's work is useful in a variety of situations, and also can be used to study the central limit theorem in some stochastic processes which have infinite variance. Examples of such processes will be studied in Chapter 6.

1.4 New convex billiards with hyperbolic behavior

Billiard models are the playground of statistical physicists and mathematicians alike, whose work focuses on the interplay between their dynamical and statistical properties. There are two main categories of chaotic billiards. The better-known ones are dispersing or semi-dispersing billiards, of which the hard-sphere gas is the prototypical example. The mechanism giving rise to chaos in these billiards is the dispersing effect, where adjacent trajectories separate exponentially fast from each other. The corresponding Jacobi field has amplitude growing exponentially as a function of the number of collisions with surfaces of positive curvature, implying that the system has a positive Lyapunov exponent [81] and is thus chaotic. Sinai billiards and Lorentz gas are essentially the only systems of interacting particles for which rigorous results, ranging from hyperbolicity to exponential decay of correlations, have been firmly established.

Many other classes of planar chaotic billiards have been found by Bunimovich [19, 21, 22], Wojtkowski [90], Markarian [69], Donnay [46], etc. These belong to the second category, which is called defocusing billiards. Here chaos is due to a mechanism different from dispersion, namely the *defocusing mechanism*. Contrary to the Sinai billiard, some boundary components of the billiard table curve inwards with respect to the particle motion, in other words they are made of convex curves. Although nearby trajectories initially focus after a collision with a convex curve, if the distance to the next collision is longer than the distance to the focal point then they eventually defocus even more. This mechanism leads to an overall expansion in phase space, again measured by a positive Lyapunov exponent [41]. Since its discovery, the mechanism of defocusing has attracted much attention in the physics community, particularly in connection with quantum chaos as well as acoustic experiments in closed chaotic cavities.

In spite of its potential appeal to a broad range of physical applications, it has, however, remained a difficult problem to prove hyperbolicity for convex billiards when the defocusing mechanism fails at each iteration. There are still only a few model of billiards that fail the defocusing mechanism and are known to be fully chaotic, see for example [28]. These allow us to observe the transitions by which the typical phase space, which is a mixture of ergodic, chaotic components, and regular KAM islands, evolves from one extreme behavior to the other.

Recently, in joint work with Jingyu Chen, Pengfei Zhang, and Hong-Kun Zhang, we discovered a new family of chaotic billiards that fails the defocusing mechanism. These tables are constructed by intersecting two circles, the smaller having radius $r = 1$ and the larger having radius $R \geq 1$, whose centers are a distance B from each other. This family, $Q(R, B)$, is characterized by the two parameters $R \geq 1$ and $B \geq 0$. This family includes many well-known tables; examples range from the circle, which is completely integrable, to the limiting case of a major arc which is closed by a straight line and is known to be ergodic and mixing. Numerical results lead us to conjecture in Chapter 7 that certain classes of the billiard tables $Q(R, B)$ are ergodic, and simulations allow us to study the transition in these tables from chaos to complete integrability.

1.5 Outline

Our main goal for this thesis is to develop central limit theorems for the return map on certain nonuniformly hyperbolic systems. We accomplish this by proving the CLT for two classes of stationary stochastic processes adapted to increasing filtrations. The main tools we use in our proofs are martingale approximation and the martingale central limit theorem, which are presented in depth by Hall and Heyde in [52]. One advantage of our methods is that we will, in many cases, be able to give explicit expressions for the scaling factor and for the variance σ^2 found in (1.3). Additionally, we find that our proposed methods are applicable to a wide variety of billiards. Our hope is that this will lead to a more unified approach to studying the statistical properties of nonuniformly hyperbolic systems with slow decay of correlations.

In Chapter 2 we provide some necessary background material in probability, martingale, and dynamical system theories which we will use throughout this document. We also state some known results for sequences of random variables and for dynamical systems under different assumptions to provide context for our current studies.

In the background material we state a central limit theorem developed by Liverani in [64] for certain classes of functions in ergodic dynamical systems. We seek to extend his results in Chapter 3 to sequence of functions in those systems.

In Chapter 4 we state and prove our main results for certain classes of stationary, adapted arrays of random variables. These are inspired by and have direct applications to the nonuniformly hyperbolic systems we subsequently study more in-depth. The arrays have weak dependence and special assumptions on their conditional expectations which arise naturally in these systems. Our assumptions lead to the non-standard scaling factor $\sqrt{n \ln n}$, and we give an explicit expression for σ^2.

In Chapter 5 we state conditions on nonuniformly hyperbolic billiards which allow us to apply the results of Chapter 4 to the return time function on these systems. Additionally, we indicate how one may prove the CLT for Hölder continuous functions given the CLT for a related induced function.

In Chapter 6 we define and introduce billiards. We study four classes of nonuniformly hyperbolic billiards and show that we can apply the results of Chapter 4 to the return time function in each of these systems. The central limit theorem has been proved in the first three tables we examine (stadia, billiards with cusps, and semi-dispersing billiards), so here we are demonstrating that our work has legitimate applications. The fourth class of table, the skewed stadia, has not been shown to satisfy the central limit theorem, and we use our methods to prove a central limit theorem on this billiard as well. Finally, we examine the example of the linked twist map, a nonuniformly hyperbolic system to which we can also apply the work of Chapter 4.

Lastly, we present joint work with Jingyu Chen, Hong-Kun Zhang, and Pengfei Zhang, to be printed in *Chaos*, in Chapter 7. We introduce a new family of two-parameter billiards, certain classes of which we conjecture are chaotic. Numerical and theoretical evidence is also supplied.

CHAPTER 2

Preliminaries

2.1 Probability theory

In this section we provide some definitions and background in probability theory. We assume basic knowledge of measure theory throughout this discussion. For a refresher on measure theory the author recommends his favorite text, Folland's Real Analysis [49]. For a more in-depth introduction to probability, see the texts by Kolmogorov [61], Shiryaev [78], and Durrett [48].

Let Ω be a set equipped with σ-algebra $\mathcal{F}$, and let $\mathbb{P}$ be a measure on $(\Omega, \mathcal{F})$. We say that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space if it is a measure space with $\mathbb{P}(\Omega) = 1$. It is often helpful to think of a set $A \in \mathcal{F}$ as a possible event, with $\mathbb{P}(A)$ being the probability of that event occuring. A measurable function $X : \Omega \to \mathbb{K}$ is called a random variable; typically we have real-valued ($\mathbb{K} = \mathbb{R}$), integer-valued ($\mathbb{K} = \mathbb{Z}$), or naturally-valued ($\mathbb{K} = \mathbb{N}$) random variables. These functions are useful in quantitatively studying possible events or the outcomes of an experiment. The expectation of a random variable X is

$$\mathbb{E}(X) = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$$

and the variance of X is

$$\mathrm{Var}(X^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \int_\Omega X^2(\omega)\, d\mathbb{P}(\omega) - \left(\int_\Omega X(\omega)\, d\mathbb{P}(\omega)\right)^2.$$

Let $\{X_i\}_{i=1}^\infty$ be a sequence of random variables. We will make use of several notions of convergence. We say that X_i converges to X almost surely and write $X_i \xrightarrow{a.s.} X$ if

$$\lim_{n\to\infty} X_i(\omega) = X(\omega)$$

for all $\omega \in \Omega$ except for possibly a set of probability zero.

The sequence X_i converges to X in probability, or $X_i \xrightarrow{p} X$, if for all $\epsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}(|X_i - X| > \epsilon) = 0.$$

We say X_i converges to X in L^p, or $X_i \xrightarrow{L^p} X$, for $p \geq 1$ if

$$\lim_{n\to\infty} \mathbb{E}\left(|X_i - X|^p\right) = 0.$$

Lastly, X_i converges to X in distribution, or $X_i \xrightarrow{d} X$, if for any Borel set $B \in \mathcal{B}(\mathbb{K})$

$$\lim_{n\to\infty} \mathbb{P}(X_i \in B) = \mathbb{P}(X \in B).$$

Two random variables X and Y are identically distributed if for all $B \in \mathcal{B}(\mathbb{K})$ we have $\mathbb{P}(X \in B) = \mathbb{P}(Y \in B)$, in particular this implies that $\mathbb{E}(X) = \mathbb{E}(Y)$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2)$. They are said to be independent if $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$ for any Borel sets A, B. This implies the equality $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ for independent random variables. A sequence of random variables $\{X_i\}_{i=1}^{\infty}$ is independent if for any $n \in \mathbb{N}$ and collection of $B_i \in \mathcal{B}(\mathbb{K})$ we have

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \{X_i \in B_i\}\right) = \prod_{i=1}^{n} \mathbb{P}(X_i \in B_i).$$

There is a great wealth of information known about the statistical properties of sequences of independent identically distributed (i.i.d.) random variables which form the basis of our studies.

2.2 Statistical theorems for sequences of i.i.d. random variables

Many astonishing theorems hold for sequences of i.i.d. random variables. Although these results are well understood their statements are not necessarily immediately apparent. We begin with the strong law of large numbers (SLLN).

Theorem 2.1 (SLLN). *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathbb{E}|X_i| < \infty$ and $\mathbb{E}(X_i) = \mu$. Then*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} \mu$$

Essentially, if one was to repeat an experiment (whose results in a given trial do not affect the results of any other trial) a large enough number of times and take the average of the results then this quantity approximates the expected value of the experiment. This is an amazing result in and of itself, but it also has a great many applications, especially for those in numerics or statistics who want to estimate the expected value of a random variable. In this situation, however, an important question remains: given n simulations of the random variable, how do you know how close your average is to the actual expectation? Fortunately the central limit theorem gives us some information on this matter.

Theorem 2.2 (Central Limit Theorem {CLT}). *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_i) = \mu$ and $Var(X_i) = \sigma^2 \in (0, \infty)$. If $S_n = X_1 + \cdots + X_n$ then*

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} Z$$

where Z has the standard normal distribution.

We write $Z \sim \mathcal{N}(0, 1)$ to mean that Z has the standard normal distribution, that is, the normal distribution with mean zero and variance one. We also note here that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then the distribution of the normal variable X is, for $x \in \mathbb{R}$,

$$\mathbb{P}(X = x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

The normal distribution is commonly referred to as the bell-shaped distribution due to the shape of its graph.

Although it may not be immediately apparent the CLT does tell us something about the SLLN. Upon rewriting, the convergence in the theorem can be interpreted as

$$\frac{S_n}{n} \sim^d \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right),$$

that is, the distribution of the average for a large number n of repetitions of the experiment can be approximated by a normal random variable with mean μ and variance σ^2/n. We recall that the variance is a probabilistic measure of how far away from the mean a single result can be; the smaller the variance the closer to the mean a given result generally is. The CLT tells us that the average of the random variable over a large number of experiments is approximately normally distributed, with mean equal to the expectation of the random variable and with a variance that decreases as we increase the number of times the experiment is repeated. Thus, for a large enough n the average S_n/n is a good approximation of the expectation μ.

Much work has been done to generalize these results for sequences of random variables which are not independent, not identically distributed, or both. In our work and applications we will see that having identically distributed random variables is a very natural condition, therefore we will focus on the work that has been done concerning sequences of identically distributed random variables which are not independent.

2.3 Limit theorems for stationary processes

A wealth of literature exists on the subject of sequences of identically distributed, stationary random variables. There are several equivalent ways to present stationary processes.

One way is to view a sequence of stationary random variables as a function of a Markov process. We assume that $\{\xi_n, n \in \mathbb{Z}\}$ denotes a stationary Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable space $(\mathcal{S}, \mathcal{G})$. The marginal distribution and the transition kernel are denoted by $\pi(A) = P(\xi_0 \in A)$ and $Q(\xi_0, A) = P(\xi_1 \in A|\xi_0)$, respectively for any $A \in \mathcal{G}$. In addition, Q denotes the operator acting via $Qf(\xi) = \int_{\mathcal{S}} f(s)Q(\xi, ds)$. Next let $L_0^2(\pi)$ be the set of functions on $\mathcal{S}$ such that $\int f^2\, d\pi < \infty$. Denote by $\mathcal{F}_k$ the σ-field generated by ξ_i, with $i < k$, $X_i = f(\xi_i)$, let $S_n = \sum_{i=0}^{n-1} X_i$ be the partial sum. We also set $\mathcal{F}_{-\infty} = \cap_{k \in \mathbb{Z}} \mathcal{F}_k$. Note that any stationary sequence $\{Y_k\}$ can be viewed as a function of a Markov process $\xi_k = \{Y_i : i \leq k\}$ for a function $g(\xi_k) = Y_k$. We also assume that these random variables are identically distributed, but may not be independent.

An alternative way to introduce the stationary processes is the following. Let $T : \Omega \to \Omega$ be an invertible transformation preserving the probability. Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{F}$ satisfying $\mathcal{F}_0 \subset T^{-1}(\mathcal{F}_0)$. We then define the increasing filtration $\{\mathcal{F}_n, n \in \mathbb{Z}\}$ by $\mathcal{F}_n = T^{-n}\mathcal{F}_0$. Let X_0 be a random variable which is $\mathcal{F}_0$-measurable. We define the stationary sequence $\{X_n, n \in \mathbb{Z}\}$ by $X_n = X_0 \circ T^n$.

There are many ways to measure the dependence between two random variables X and Y and there are associated CLTs for many of these notions. In what follows we will introduce a few of these ideas. For a deeper treatment of this subject see the survey articles by Bradley [16] and Withers [88, 89].

We first introduce the following mixing coefficients. Let $\sigma(X, Y, Z)$ be the σ-algebra generated by random variables X, Y, and Z.

Definition 1. *For any two σ-algebras $\mathcal{A}$ and $\mathcal{B}$, define the strong mixing coefficient $\alpha(\mathcal{A}, \mathcal{B})$ as*

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{A}, B \in \mathcal{B}\}$$

For stationary process $\{X_k\}_{k\in\mathbb{Z}}$, we also define $\alpha(n) := \alpha(\mathcal{F}_0, \sigma(X_n))$. We say that the sequence $\{X_i\}$ is strong mixing if $\alpha(n) \to 0$ as $n \to \infty$.

This indicates that for a strong mixing process the random variables become less and less dependent; in a sense they limit to independence. The CLT has been established for strong mixing sequences which satisfy a few extra assumptions. The following theorem was proved by Ibragimov in 1962 [57].

Theorem 2.3. *Assume that $\{X_i\}_{i=1}^{\infty}$ is an identically distributed and strong mixing sequence of random variables, with $\mathbb{E}(X_n) = 0$, $\mathbb{E}(X_n^{2+\delta}) < \infty$ for some $\delta > 0$, and $\sum_{k\geq 1} \alpha(k)^{\delta/(2+\delta)} < \infty$. Then $\{X_n\}$ satisfies the central limit theorem. More precisely, denote $S_n = X_1 + \cdots + X_n$. Then the limit*

$$\sigma^2 = \lim_{n\to\infty} \frac{\mathbb{E}(S_n^2)}{n} \tag{2.1}$$

exists, and if $\sigma^2 \neq 0$ then

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1) \tag{2.2}$$

and

$$\sigma^2 = \mathbb{E}(X_1^2) + 2\sum_{k=1}^{\infty} \mathbb{E}(X_1 X_{1+k}). \tag{2.3}$$

This result is quite amazing. Even though there is dependence among the random variables, the CLT still holds under an assumption on some higher moment and the speed of the strong mixing.

In some applications it is more natural to use a different measure of dependence. Recall that if X and Y are independent then it is true that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. If this property holds we say that X and Y are uncorrelated. The covariance of two random variables X and Y is

$$\mathrm{Cov}(X,Y) = |\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)|$$

and the correlation of two random variables with finite variance is

$$\mathrm{corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}.$$

These definitions also hold for square-integrable functions f and g. Now, for any two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ we define the maximal correlation between them to be

$$\rho(\mathcal{F}_1, \mathcal{F}_2) = \sup |\mathrm{corr}(f,g)|$$

where the supremum is taken over all real-valued $f \in L^2(\mathcal{F}_1)$ and $g \in L^2(\mathcal{F}_2)$.

Suppose that $\{X_i\}_{i=1}^{\infty}$ is a sequence of identically distributed random variables. For each $n \in \mathbb{N}$ the dependence coefficient is defined as

$$\rho(n) = \rho\left(\sigma(X_1), \sigma(X_{n+1}, X_{n+2}, \ldots)\right).$$

The sequence $\{X_i\}$ is said to be ρ-mixing if $\rho(n) \to 0$ as $n \to \infty$. In other words, a sequence is ρ-mixing if functions which are square integrable in the sigma algebras generated by its random variables become uncorrelated as the time step between those random variables becomes arbitrarily large.

Before stating a CLT for ρ-mixing sequences we need one further definition. Given a function $g : [0, \infty) \to [0, \infty)$ such that $g(x) > 0$ for all x sufficiently large, we say that $g(x)$ is slowly varying as $x \to \infty$ if for all $t > 0$ we have $g(tx)/g(x) \to 1$ as $x \to \infty$.

Theorem 2.4 (Ibragimov (1975) [58])**.** *Let $\{X_i\}_{i=1}^{\infty}$ be a ρ-mixing, identically distributed sequence of random variables satisfying $\mathbb{E}(X_i) = 0$ and $\mathbb{E}(X_i^2) < \infty$. Define $S_n = \sum_{i=1}^{n} X_i$ and $\sigma_n^2 = \mathbb{E}(S_n^2)$. Then either $\sup_n \sigma_n^2 < \infty$ or $\sigma_n^2 = nh(n)$, where $h(n)$ is a slowly varying function. If, in addition, $\mathbb{E}|X_i|^{2+\delta} < \infty$ for some $\delta > 0$ and $\sigma_n^2 \to \infty$, then*

$$\frac{S_n}{\sigma_n} \xrightarrow{d} \mathcal{N}(0,1).$$

Ibragimov showed that a CLT holds for stationary, ρ-mixing sequences with minimal assumptions on moments. However, this theorem does not give us a very specific expression for the scaling factor σ_n. Fortunately, in the same paper he proved a statement that looks more like the classical CLT.

Theorem 2.5 (Ibragimov (1975) [58])**.** *Let $\{X_i\}_{i=1}^{\infty}$ be a stationary sequence of random variables satisfying $\mathbb{E}(X_i) = 0$, $\mathbb{E}(X_i^2) < \infty$, and $\sum \rho(2^n) < \infty$. Define $S_n = \sum_{i=1}^{n} X_i$ and $\sigma_n^2 = \mathbb{E}(S_n^2)$. Then the sequence has continuous spectral density $f(\lambda)$, and if $f(0) \neq 0$, then*

$$\sigma_n^2 = 2\pi f(0) n (1 + o(1)),$$

and

$$\frac{S_n}{\sqrt{2\pi f(0) n}} \xrightarrow{d} \mathcal{N}(0,1).$$

The convergence here is of the form seen in the classical CLT, but we have to make some stronger assumptions about the speed of the ρ-mixing and the spectrum of the sequence. Relaxing the condition of independence is one of the ways the CLT has been generalized.

We can now address a situation much closer to our applications. We would like to study certain sequences of identically distributed random variables which have infinite variance. In particular we would like these sequences to be ρ-mixing. In chapter six we will show that these kinds of sequences arise in billiard dynamics and our work in chapters four and five will be used to demonstrate that the CLT holds for certain functions on several types of billiard. One inspiration for some of our results is the following theorem.

Theorem 2.6 (Bradley (1988) [15])**.** *Let $\{X_i\}_{i=1}^{\infty}$ be a stationary sequence of real-valued random variables satisfying $\mathbb{E}(X_i) = 0$, $\rho(1) < 1$, $\sum \rho(2^n) < \infty$, and*

$$\lim_{c\to\infty} c^2 \mathbb{P}(|X_1| > c)/\mathbb{E}(X_1^2 \mathbf{1}_{\{|X_1|\le c\}}) = 0.$$

Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers with $a_n \to \infty$ as $n \to \infty$, such that

$$\frac{S_n}{a_n} \xrightarrow{d} \mathcal{N}(0,1).$$

We note here that this theorem acts as an extension of Theorem 2.4. Bradley makes no assumption about the variance of the random variables, in fact, he states in his paper that if the variance is finite then his theorem essentially reduces to Ibragimov's. However, there is no follow-up theorem in this case which gives the form of a_n under stronger assumptions. It turns out that martingale approximation is more powerful and leads to stronger results which provide an expression for the sequence a_n. To explain the main idea we will need to develop the necessary machinery for this technique in the following section.

2.4 The martingale central limit theorem

Here we will introduce conditional probability, expectation and the concept of a martingale. For some thorough treatments of these subjects see the texts of Breiman [17], Durrett [48], and Hall and Heyde [52].

If two events or random variables are dependent we can consider the *conditional probability* of one given the other. Classically, for sets A and B we have that the conditional probability of B

occurring given A is $\mathbb{P}(B|A) = \mathbb{P}(A \cap B)/\mathbb{P}(A)$. This concept is extendable to random variables or σ-algebras but we will not delve into that here.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X a random variable on Ω, and $\mathcal{F}_0$ a sub-σ-algebra of $\mathcal{F}$. Then the *conditional expectation of X given $\mathcal{F}_0$*, denoted $\mathbb{E}(X|\mathcal{F}_0)$, is a random variable satisfying

1. $\mathbb{E}(X|\mathcal{F}_0)$ is $\mathcal{F}_0$-measurable.

2. for all A in $\mathcal{F}_0$, $\int_A X d\mathbb{P} = \int_A \mathbb{E}(X|\mathcal{F}_0) d\mathbb{P}$.

Conditional expectation enjoys a few nice properties that we will make use of. If X is $\mathcal{F}_0$-measurable, then $\mathbb{E}(X|\mathcal{F}_0) = X$. Similarly, if X is $\mathcal{F}_0$-measurable and $\mathbb{E}|Y|, \mathbb{E}|XY| < \infty$ then $\mathbb{E}(XY|\mathcal{F}_0) = X\mathbb{E}(Y|\mathcal{F}_0)$. It is a fact that $\mathbb{E}(X|\{\emptyset, \Omega\}) = \mathbb{E}(X)$. If $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{G}$, then

$$\mathbb{E}\left(\mathbb{E}(X|\mathcal{F})|\mathcal{G}\right) = \mathbb{E}\left(\mathbb{E}(X|\mathcal{G})|\mathcal{F}\right) = \mathbb{E}(X|\mathcal{F}).$$

Consequently, we also have that $\mathbb{E}\left(\mathbb{E}(X|\mathcal{F})\right) = \mathbb{E}(X)$.

Conditional expectations are often hard to get a handle on, but there are some cases where explicit expressions are possible. One instance that we will employ later on is the following: suppose $\Omega_1, \Omega_2, \ldots$ is an infinite partition of Ω into disjoint sets, each of which has positive probability, and let $\mathcal{F}_0 = \sigma(\Omega_1, \Omega_2, \ldots)$ be the σ-algebra generated by these sets. Then

$$\mathbb{E}(X|\mathcal{F}_0) = \sum_{i=1}^{\infty} \frac{\mathbb{E}(X;\Omega_i)}{\mathbb{P}(\Omega_i)} \mathbf{1}_{\Omega_i}$$

where $\mathbb{E}(X;\Omega_i) = \int_{\Omega_i} X d\mathbb{P}$. In this situation the conditional expectation is a simple function and, we will see, is something that can be computed explicitly in some cases.

Suppose $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is an increasing sequence of sub-σ-algebras of $\mathcal{F}$, and $\{X_i\}_{i=1}^{\infty}$ is a sequence of random variables on Ω satisfying

1. X_i is measurable with respect to $\mathcal{F}_i$, in particular, $\mathbb{E}(X_i|\mathcal{F}_i) = X_i$,

2. $\mathbb{E}|X_i| < \infty$,

3. $\mathbb{E}(X_{i+1}|\mathcal{F}_i) = X_i$ a.s. for all $i \in \mathbb{N}$.

Then $\{(X_i, \mathcal{F}_i)\}_{i=1}^{\infty}$ is said to be a *martingale*. If the third condition is replaced by $\mathbb{E}(X_{i+1}|\mathcal{F}_i) \geq X_i$ a.s. then $\{(X_i, \mathcal{F}_i)\}_{i=1}^{\infty}$ is a *submartingale* and if $\mathbb{E}(X_{i+1}|\mathcal{F}_i) \leq X_i$ a.s. it is a *supermartingale*. Lastly, if $\mathbb{E}(X_{i+1}|\mathcal{F}_i) = 0$ then we have a *martingale difference*.

A *reverse* or *backwards martingale* is adapted to a decreasing sequence of σ-algebras $\mathcal{F}_i \subset \mathcal{F}_{i-1}$, and satisfies the first two properties above, but instead of property three we have $\mathbb{E}(X_{i-1}|\mathcal{F}_i) = X_i$ almost surely.

The martingale is a concept which, as is often the case in probability theory, has its origins in gambling. In this case a martingale can be thought of as a fair game where X_i is the fortune of a gambler after i games and $\mathcal{F}_i$ the information contained in the first i games. The third condition tells us that the gambler can not expect to gain or lose any money playing this game. On the other hand, a submartingale is a game the gambler expects to make money on, and a supermartingale is one in which he expects his fortunes to dwindle.

Suppose for all $n \geq 1$ that $\{(X_{n,i}, \mathcal{F}_{n,i})\}_{i=1}^{k_n}$ is a martingale for some constant k_n depending on n. We say that $\{(X_{n,i}, \mathcal{F}_{n,i})\}_{i=1}^{k_n}$ is a *martingale array.* If we set $Z_{n,i} = X_{n,i} - X_{n,i-1}$ the we call $\{Z_{n,i}\}$ a *martingale difference array.* There exist beautiful limit theorems for martingale difference arrays and we will present two versions of a martingale CLT here. The first can be found in Hall and Heyde's text [52].

Theorem 2.7 (Martingale Central Limit Theorem)**.** *Let $\{(X_{n,i}, \mathcal{F}_{n,i}), 1 \leq i \leq k_n, n \geq 1\}$ be a zero-mean, square-integrable martingale array with differences $Z_{n,i}$, and let σ^2 be an a.s. finite random variable. Suppose that*

$$\max_i |Z_{n,i}| \xrightarrow{p} 0,$$

$$\sum_{i=1}^{k_n} Z_{n,i}^2 \xrightarrow{p} \sigma^2,$$

$$\mathbb{E}\left(\max_i Z_{n,i}^2\right) \quad \textit{is bounded in } n,$$

and the σ-algebras are nested, that is, $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}$ for $1 \leq i \leq k_n$, $n \geq 1$.
Then $S_{n,k_n} = \sum_i Z_{n,i} \xrightarrow{d} Z$, where the random variable Z has characteristic function $\mathbb{E}\left(e^{-\sigma^2 t^2/2}\right)$.

This result is a powerful one, but is a little more than we will need for our work. Instead we will use a strong form of a corollary in [52], presented by Sethuraman and Varadhan in their 2008 paper [77].

Theorem 2.8 (Martingale Central Limit Theorem)**.** *Let $\{(X_{n,i}, \mathcal{F}_{n,i}), 0 \leq i \leq n, n \geq 1\}$ be a martingale relative to the nested family $\mathcal{F}_{n,i} \subset \mathcal{F}_{n,i+1}$ with $X_{n,0} = 0$. Let $Z_{n,i} = X_{n,i} - X_{n,i-1}$ be their differences. Suppose that*

$$\max_{1 \leq i \leq n} \|Z_{n,i}\|_{L^\infty} \to 0$$

and

$$\sum_{i=1}^{n} \mathbb{E}\left(Z_{n,i}^2 | \mathcal{F}_{n,i-1}\right) \xrightarrow{L^2} 1.$$

Then

$$X_{n,n} \xrightarrow{d} \mathcal{N}(0,1).$$

Note that the main difference between these two versions of the martingale CLT is the second assumption. In the second case we are examining the convergence of the sum of conditional expectations on the squared differences, rather than the sum of the squared differences themselves. We are also requiring a stronger form of convergence, in L^2 instead of in probability. Even more importantly, this sum converges to a constant as opposed to a random variable. This will be enough for our theory and applications.

The martingale CLT is a very natural way to handle the CLT for stochastic processes generated by dynamical systems, including the nonuniformly hyperbolic systems we examine later. Before moving on to our main theorems, however, it will be beneficial to discuss the definitions and properties of dynamical systems in general. In addition we will state a CLT for dynamical systems which we generalize in chapter three.

2.5 Dynamical systems

In this section we give a basic introduction to dynamical systems and their statistical properties. For an exhaustive treatment of the subject see Katok and Hasselblatt's classic text [59].

A dynamical system (X, T) consists of a measurable space X and a measurable map $T : X \to X$. A measurable function $\phi : X \to \mathbb{C}$ (or $\mathbb{R}$) is called an *observable* on X. After equipping the dynamical system (X, T) with a probability measure μ, one can consider the sequence $\{\phi \circ T^i\}_{i=0}^{\infty}$ as a sequence of random variables. Because these systems are deterministic this sequence is not independent, however as we have seen in previous sections the CLT can still hold for sequences with dependence which is, in some sense, weak enough.

This raises the question: how does one choose which measure to associate to the dynamical system? Ideally we would like our sequence to be identically distributed. To accomplish this we want our measure μ to be *invariant* with respect to T, that is (assuming X is equipped with σ-algebra $\mathcal{F}$) for any $A \in \mathcal{F}$ we have $\mu(T^{-1}A) = \mu(A)$. This implies in particular that for Borel sets $B \in \mathbb{C}$ (or $\mathbb{R}$) we have $\mu(\phi(x) \in B) = \mu(\phi(Tx) \in B)$, and that $\mathbb{E}(\phi) = \mathbb{E}(\phi \circ T^k)$ for any $k > 0$. In fact, an equivalent definition for the invariance of μ is that for all $\phi \in L^1(X, \mu)$ we have $\mathbb{E}(\phi) = \mathbb{E}(\phi \circ T)$. The existence and uniqueness of invariant measures is a deep subject in and of itself, and while it is not guaranteed that a given dynamical system has an invariant measure we will give an explicit expression for an invariant measure of the systems we study in chapter six.

Suppose that (X, T, μ) is a dynamical system with a T-invariant probability measure. We can now begin to consider the statistical properties of the system. We say that a dynamical system is *ergodic* if all invariant sets $A = T^{-1}A$ have either zero measure or full measure. Interestingly, being ergodic is equivalent to the system satisfying the strong law of large numbers for any $f \in L^1(\mu)$, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \mathbb{E}(f)$$

for almost every $x \in X$. We would like to study the CLT, and as we have seen in previous sections we generally require at least a decay of correlations in order for the theorem to hold. In dynamical systems this property is called *mixing*. We say a system is mixing if for all $f, g \in L^2(X, \mu)$ we have

$$\lim_{n\to\infty} \mathbb{E}(f \circ T^n \cdot g) = \mathbb{E}(f)\,\mathbb{E}(g)$$

or, in other words, if

$$\lim_{n\to\infty} \mathrm{Cov}(f \circ T^n, g) = 0.$$

This property alone is not a sufficient assumption for the CLT in most dynamical systems. The speed of mixing is also quite an important property of the system. In the following section we present a CLT for dynamical systems which we seek to extend in chapter three.

2.6 Central limit theorem for deterministic systems

The main theorem of this section, as well as its proof, can be found in Liverani's 1996 paper [64]. We let $(X, \mathcal{F}, T, \mathbb{P})$ be an ergodic dynamical system equipped with T-invariant probability measure $\mathbb{P}$ and σ-algebra $\mathcal{F}$.

For each $\phi \in L^2(X)$ we define the map $\hat{T} : L^2(X) \to L^2(X)$ by

$$\hat{T}\phi = \phi \circ T,$$

and let $\hat{T}^*: L^2(X) \to L^2(X)$ be its dual. Consider a sub-σ-algebra $\mathcal{F}_0$ of $\mathcal{F}$ and define $\mathcal{F}_i = T^{-i}\mathcal{F}_0$ for $i \in \mathbb{Z}$, then the following theorem holds.

Theorem 2.9 (CLT for Deterministic Systems [64]). *If $\mathcal{F}_i \subset \mathcal{F}_{i-1}$ and for each $\phi \in L^\infty(X)$ we have*

$$\mathbb{E}(\hat{T}\hat{T}^*\phi|\mathcal{F}_1) = \mathbb{E}(\phi|\mathcal{F}_1),$$

then for each $f \in L^\infty(X)$, $\mathbb{E}(f) = 0$ and $\mathbb{E}(f|\mathcal{F}_0) = f$, such that

(a) $\sum_{i=0}^{\infty} |\mathbb{E}(f\hat{T}^i f)| < \infty$,

(b) the series $\sum_{i=0}^{\infty} \mathbb{E}(\hat{T}^{*i} f|\mathcal{F}_0)$ *converges absolutely almost surely,*

then

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \hat{T}^i f \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 \leq -\mathbb{E}(f^2) + 2\sum_{i=0}^{\infty} \mathbb{E}(f\hat{T}^i f)$. *In addition, $\sigma = 0$ if and only if there exists a $\mathcal{F}_0$-measurable function g such that*

$$\hat{T}f = \hat{T}g - g.$$

Finally, if the series in (b) converges in $L^1(X)$, then $\sigma^2 = -\mathbb{E}(f^2) + 2\sum_{i=0}^{\infty} \mathbb{E}(f\hat{T}^i f)$.

Note that in this theorem we require that mixing is fast enough that the series $\sum_{i=0}^{\infty} |\mathbb{E}(f\hat{T}^i f)|$ converges.

The main idea of the proof is to decompose $\hat{T}^i f$ into two parts: one term which is a backwards martingale difference and a second term which, upon taking a sum, is negligibly small in distribution. It is then enough to show that the CLT holds for the martingale. We will use the same ideas in our proofs, so more details may be found in chapter three or in Liverani's paper.

CHAPTER 3

Classical Central Limit Theorems for Dynamical Systems

In this chapter we focus on expanding Liverani's Theorem 1.1 in [64] to several types of dynamical systems. We will begin by translating his statement to instead include increasing σ-algebras, so that it more closely resembles the systems in our applications. This will also have the effect that the proof will involve forward rather than backward martingales.

We first illustrate the main idea. Let $\{X_n\}$ be any stationary process with zero mean. We can often succeed in writing $X_n = \xi_{n+1} + \eta_{n+1}$ where ξ_n is a martingale difference and η_n is negligible, in the sense that

$$\mathbb{E}[(\sum_{i=1}^{n} \eta_i)^2] = o(n) \tag{3.1}$$

Then the central limit theorem for $\{X_n\}$ can be deduced from that of $\{\xi_n\}$ using Theorem 2.7 in Chapter 2. A cheap way to prove (3.1) is to establish that $\eta_n = Z_n - Z_{n+1}$ for some stationary square integrable sequence $\{Z_n\}$. Then $\sum_{i=1}^{n} \eta_i$ telescopes and the needed estimate is obvious. Here is a way to construct Z_n from X_n so that $X_n - (Z_{n+1} - Z_n)$ is a martingale difference.

We define $Z_n = \sum_{j=0}^{\infty} \mathbb{E}(X_{n+j}|\mathcal{F}_n)$. If one can guarantee that the series converges in L^2 then Z_n exists and is square integrable. Then

$$Z_n = \mathbb{E}(Z_{n+1}|\mathcal{F}_n) + X_n$$

or equivalently,

$$X_n = Z_n - \mathbb{E}(Z_{n+1}|\mathcal{F}_n) = (Z_n - Z_{n+1}) + (Z_{n+1} - \mathbb{E}(Z_{n+1}|\mathcal{F}_n)) = \eta_{n+1} + \xi_{n+1}$$

where $\xi_{n+1} = Z_{n+1} - \mathbb{E}(Z_{n+1}|\mathcal{F}_n)$, $\eta_{n+1} = Z_n - Z_{n+1}$. It is easy to see that $E(\xi_{n+1}|\mathcal{F}_n) = 0$, thus $\{\xi_n\}$ is a martingale difference.

3.1 A forward version of Liverani's theorem

Let $(X, \mathcal{F}, T, \mathbb{P})$ be an ergodic dynamical system equipped with T-invariant probability measure $\mathbb{P}$ and σ-algebra $\mathcal{F}$. For each $\phi \in L^2(X)$ we define the map $\hat{T} : L^2(X) \to L^2(X)$ by

$$\hat{T}\phi = \phi \circ T,$$

and let $\hat{T}^* : L^2(X) \to L^2(X)$ be its dual. Consider a sub-σ-algebra $\mathcal{F}_0$ of $\mathcal{F}$ and define $\mathcal{F}_i = T^{-i}\mathcal{F}_0$ for $i \in \mathbb{Z}$, then we have the following theorem.

Theorem 3.1. *If $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ then for each $f \in L^\infty(X)$, $\mathbb{E}(f) = 0$, and $\mathbb{E}(f|\mathcal{F}_0) = f$ such that*

(a) $\sum_{n=0}^{\infty} \left|\mathbb{E}(f\hat{T}^n f)\right| < \infty$,

(b) $\sum_{n=1}^{\infty} \mathbb{E}(\hat{T}^n f|\mathcal{F}_0)$ *converges absolutely almost surely,*

we have

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \hat{T}^i f \to \mathcal{N}(0, \sigma^2)$$

where $\sigma^2 \leq \mathbb{E}(f^2) + 2\sum_{n=1}^{\infty} \mathbb{E}(f\hat{T}^n f)$*. Furthermore,* $\sigma = 0$ *if and only if there exists an* $\mathcal{F}_0$*-measurable function* g *such that* $\hat{T}f = g - \hat{T}g$*. Lastly, if the sequence in (b) converges in* $L^1(X)$ *then* $\sigma^2 = \mathbb{E}(f^2) + 2\sum_{n=1}^{\infty} \mathbb{E}(f\hat{T}^n f)$*.*

The proof of this theorem will follow Liverani's closely. Most adjustments are made to reflect the differences in the statements of each theorem, otherwise the methods employed are quite similar. It should also be noted that for a given function $f \in L^\infty(X)$ there is a very natural sub-σ-algebra $\mathcal{F}_0$ such that $\mathbb{E}(f|\mathcal{F}_0) = f$ and $\mathcal{F}_i \subset \mathcal{F}_{i+1}$, namely $\mathcal{F}_0 = \sigma(\ldots, f \circ T^{-2}, f \circ T^{-1}, f)$. Whether or not condition (b) is satisfied will of course depend on the system being studied.

Proof of Theorem 3.1. The main idea of this proof is to use a martingale difference approximation and a martingale CLT. We seek $Y_i \in L^2(X)$ such that Y_i is a martingale difference with respect to $\{F_i\}_{i=0}^{\infty}$, that is, Y_i is $\mathcal{F}_i$-measurable and $\mathbb{E}(Y_i|\mathcal{F}_{i-1}) = 0$. Furthermore, we wish to find an $\mathcal{F}_0$-measurable function g that is finite almost everywhere such that

$$\hat{T}^i f = Y_i + \hat{T}^{i-1} g - \hat{T}^i g \tag{3.2}$$

for all $i > 0$. Summing and dividing by $\sqrt{n}$ makes the above expression

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \hat{T}^i f = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} Y_i + \frac{1}{\sqrt{n}} \left[\hat{T}^n g - g\right].$$

Noting that g is finite almost everywhere we know that $\frac{1}{\sqrt{n}}\left[\hat{T}^n g - g\right]$ converges to zero in probability as n limits to infinity. Therefore the central limit theorem for the martingale differences Y_i implies the CLT for the observable f. Our goal now is to show the CLT for Y_i.

Before proceeding we must find an expression for g. Taking the conditional expectation of (3.2) with respect to $\mathcal{F}_{i-1}$ yields

$$\mathbb{E}(\hat{T}^i f|\mathcal{F}_{i-1}) = \mathbb{E}(\hat{T}^{i-1} g|\mathcal{F}_{i-1}) - \mathbb{E}(\hat{T}^i g|\mathcal{F}_{i-1})$$

since Y_i is a martingale difference. From the definition of $\mathcal{F}_i$ we have that, for each $\phi \in L^1(X)$,

$$\mathbb{E}(\hat{T}^i \phi|\mathcal{F}_i) = \hat{T}^i \mathbb{E}(\phi|\mathcal{F}_0)$$

for all $i > 0$. This implies that

$$\hat{T}^{i-1}\mathbb{E}(\hat{T} f|\mathcal{F}_0) = \hat{T}^{i-1}\mathbb{E}(g|\mathcal{F}_0) - \hat{T}^{i-1}\mathbb{E}(\hat{T} g|\mathcal{F}_0),$$

but in particular,

$$\mathbb{E}(\hat{T} f|\mathcal{F}_0) = g - \mathbb{E}(\hat{T} g|\mathcal{F}_0).$$

One can check that $g = \sum_{n=1}^{\infty} \mathbb{E}(\hat{T}^n f|\mathcal{F}_0)$ is a solution to this equation. By assumption this sum converges absolutely almost surely. Now that we have an expression for g, the martingale differences Y_i are defined by (3.2). By construction the Y_i are $\mathcal{F}_i$-measurable and will satisfy $\mathbb{E}(Y_i|\mathcal{F}_{i-1}) = 0$. It should also be noted that we have $Y_i = \hat{T}^{i-1} Y_1$.

We now would like to use the following [71]:

Theorem 3.2. *Let $\{Y_n\}_{n=1}^{\infty}$ be a stationary, ergodic, martingale difference with respect to the filtration $\{\mathcal{F}_n\}_{n=1}^{\infty}$. If $Y_1 \in L^2(X)$, then $\sigma^2 = \mathbb{E}(Y_1^2)$ and the CLT holds.*

It only remains to show that $Y_1 \in L^2(X)$ and to calculate the appropriate bounds for σ^2. Liverani notes in his proof that if g converges in $L^2(X)$ then $Y_1 \in L^2(X)$ is immediate. However, as he goes on to show, Y_i satisfies the assumptions of this theorem even with our fairly weak assumptions on g. The method we employ here is again based on his techniques.

We construct a sequence of martingale differences $Y_i(\lambda)$ that approximate Y_i. For $\lambda > 1$ we would like $Y_i(\lambda)$ to be $\mathcal{F}_i$-measurable and $\mathbb{E}(Y_i(\lambda)|\mathcal{F}_{i-1}) = 0$. We want

$$\hat{T}^i f = Y_i(\lambda) + \hat{T}^{i-1} g(\lambda) - \lambda^{-1}\hat{T}^i g(\lambda)$$

for all $i > 0$ and all $\lambda > 1$. One can check that the function $g(\lambda)$ is

$$g(\lambda) = \sum_{n=1}^{\infty} \lambda^{-(n-1)} \mathbb{E}(\hat{T}^n f | \mathcal{F}_0)$$

and consequently that $g(\lambda) \in L^2(X)$. Furthermore $\lim_{\lambda \to 1} g(\lambda) = g$ almost surely. This implies that $\lim_{\lambda \to 1} Y_i(\lambda) = Y_i$ almost surely as well. Now,

$$\begin{aligned}
\mathbb{E}(Y_1(\lambda)^2) &= \mathbb{E}([\hat{T}f + \lambda^{-1}\hat{T}g(\lambda) - g(\lambda)]^2) \\
&= \mathbb{E}([\hat{T}f + \lambda^{-1}\hat{T}g(\lambda)][\hat{T}f + \lambda^{-1}\hat{T}g(\lambda) - g(\lambda)]) - \mathbb{E}(g(\lambda)[\hat{T}f + \lambda^{-1}\hat{T}g(\lambda) - g(\lambda)]).
\end{aligned}$$

Note that

$$\mathbb{E}(g(\lambda)[\hat{T}f + \lambda^{-1}\hat{T}g(\lambda) - g(\lambda)]|\mathcal{F}_0) = g(\lambda)\mathbb{E}(Y_1|\mathcal{F}_0) = 0.$$

Taking expectations yields $\mathbb{E}(g(\lambda)[\hat{T}f + \lambda^{-1}\hat{T}g(\lambda) - g(\lambda)]) = 0$, so we have

$$\begin{aligned}
\mathbb{E}(Y_1(\lambda)^2) &= \mathbb{E}([\hat{T}f + \lambda^{-1}\hat{T}g(\lambda)][\hat{T}f + \lambda^{-1}\hat{T}g(\lambda) - g(\lambda)]) \\
&= \mathbb{E}\left((\hat{T}f)^2\right) + 2\lambda^{-1}\mathbb{E}(\hat{T}f\hat{T}g(\lambda)) - \mathbb{E}(\hat{T}fg(\lambda)) - \lambda^{-1}\mathbb{E}(g(\lambda)\hat{T}g(\lambda)) + \lambda^{-2}\mathbb{E}([\hat{T}g(\lambda)]^2) \\
&= \mathbb{E}(f^2) + 2\lambda^{-1}\mathbb{E}(fg(\lambda)) - (1 - \lambda^{-2})\mathbb{E}(g(\lambda)^2) \\
&\leq \mathbb{E}(f^2) + 2\sum_{n=1}^{\infty} \lambda^{-n}\mathbb{E}(f\hat{T}^n f) \\
&\leq \mathbb{E}(f^2) + 2\sum_{n=1}^{\infty} \mathbb{E}(f\hat{T}^n f).
\end{aligned}$$

The above inequality holds thanks to T-invariance and one other calculation. We have that

$$\begin{aligned}
\mathbb{E}(\hat{T}fg(\lambda) + \lambda^{-1}g(\lambda)\hat{T}g(\lambda)|\mathcal{F}_0) &= g(\lambda)\mathbb{E}(\hat{T}f|\mathcal{F}_0) + \lambda^{-1}g(\lambda)\mathbb{E}(\hat{T}g(\lambda)|\mathcal{F}_0) \\
&= g(\lambda)\mathbb{E}(\hat{T}f|\mathcal{F}_0) + \lambda^{-1}g(\lambda)\left[\lambda(g(\lambda) - \mathbb{E}(\hat{T}f|\mathcal{F}_0))\right] \\
&= g(\lambda)^2.
\end{aligned}$$

The desired bound for σ^2 then follows from

$$\mathbb{E}(Y_1^2) = \mathbb{E}(\liminf_{\lambda \to 1} Y_1(\lambda)^2) \leq \liminf_{\lambda \to 1} \mathbb{E}(Y_1(\lambda)^2) \leq \mathbb{E}(f^2) + 2\sum_{n=1}^{\infty} \mathbb{E}(f\hat{T}^n f).$$

The previous theorem implies that the CLT holds for the martingale differences, and due to our earlier considerations this implies the CLT for the dynamical system. We also find that since

$$\mathbb{E}(Y_1^2) = \mathbb{E}([\hat{T}f + \hat{T}g - g]^2)$$

this implies that $\sigma = 0$ if and only if $\hat{T}f = g - \hat{T}g$.

The proof of the last claim, that $\sigma^2 = \mathbb{E}(f^2) + 2\sum_{n=1}^{\infty} \mathbb{E}(f\hat{T}^n f)$ if g converges in $L^1(X)$, is largely technical and not particularly illustrative. The argument is based largely on controlling the size of $(1-\lambda^{-2})\mathbb{E}(g(\lambda)^2)$ and is fairly independent of the differences in the assumptions of our theorem compared to Liverani's. We therefore refer the reader to [64] for the proof of this claim. □

3.2 CLT for stationary ergodic martingale difference arrays

In order to generalize Liverani's theorem we will also need a new version of Theorem 3.2. We want to maintain the assumptions of stationarity and ergodicity while having a difference array rather than a difference sequence.

Theorem 3.3. *Suppose for $n \geq 1$ that $\{Z_{n,i}\}_{i=0}^{n-1}$ is stationary ergodic sequence adapted to an increasing filtration $\{\mathcal{F}_{n,i}\}_{i=0}^{n-1}$ such that $\lim_{n\to\infty} \mathbb{E}(Z_{n,0}) = \sigma^2 < \infty$ and $\mathbb{E}(Z_{n,i}|\mathcal{F}_{n,i-1}) = 0$. Then*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} Z_{n,i} \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

As with Theorem 3.2, this result is a corollary of the martingale CLT presented in Hall and Heyde [52], which we stated in Theorem 2.7.

Proof. Define $\hat{Z}_{n,i} = Z_{n,i}/\sqrt{n}$. Then $\{\hat{Z}_{n,i}\}$ is a martingale difference array with respect to $\{\mathcal{F}_{n,i}\}$ and it satisfies the CLT by Theorem 2.7 (or Theorem 2.8). In particular, we will show that

$$\sum_{i=1}^{n} \hat{Z}_{n,i}^2 \xrightarrow{L^2} \sigma^2.$$

We have, by ergodicity,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left(\left[\sum_{i=1}^{n} Z_{n,i}^2 - \sigma^2\right]^2\right) \\
&= \lim_{n\to\infty} \mathbb{E}\left(\left[\frac{1}{n}\sum_{i=1}^{n} Z_{n,i}^2\right]^2 - \frac{2\sigma^2}{n}\sum_{i=1}^{n} Z_{n,i}^2 + \sigma^4\right) \\
&= \mathbb{E}\left(\lim_{n\to\infty}\left(\left[\frac{1}{n}\sum_{i=1}^{n} Z_{n,i}^2\right]^2 - \frac{2\sigma^2}{n}\sum_{i=1}^{n} Z_{n,i}^2 + \sigma^4\right)\right) \\
&= \mathbb{E}(\sigma^4 - 2\sigma^4 + \sigma^4) = 0.
\end{aligned}$$

□

The above will be used in the following section to prove an extension of Liverani's theorem.

3.3 CLT for sequences of functions in deterministic systems

We now present a central limit theorem for sequences of functions in dynamical systems. The setup is the same as in Section 3.1. Let $(X, \mathcal{F}, T, \mathbb{P})$ be an ergodic dynamical system equipped with T-invariant probability measure $\mathbb{P}$ and σ-algebra $\mathcal{F}$. For each $\phi \in L^2(X)$ define the map $\hat{T} : L^2(X) \to L^2(X)$ by

$$\hat{T}\phi = \phi \circ T,$$

and let $\hat{T}^* : L^2(X) \to L^2(X)$ be its dual. Consider sub-σ-algebras $\mathcal{F}_{n,0}$ of $\mathcal{F}$ and define $\mathcal{F}_{n,i} = T^{-i}\mathcal{F}_{n,0}$ for $i \in \mathbb{Z}$, then the following is true.

Theorem 3.4. *If $\mathcal{F}_{n,i} \subset \mathcal{F}_{n,i+1}$ then for each sequence of functions $\{f_n\}_{n\geq 1}$ with $f_n \in L^\infty(X)$, $\mathbb{E}(f_n) = 0$, $\mathbb{E}(f_n|\mathcal{F}_{n,0})$ such that*

(a) $\lim_{n\to\infty} \mathbb{E}(f_n^2) < \infty$,

(b) $\lim_{n\to\infty} \sum_{i=1}^{\infty} \left|\mathbb{E}(f_n \hat{T}^i f_n)\right| < \infty$,

(c) $\sum_{i=1}^{\infty} \mathbb{E}(\hat{T}^i f_n|\mathcal{F}_{n,0})$ *converges absolutely almost surely,*

then we have

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \hat{T}^i f_n \to \mathcal{N}(0, \sigma^2)$$

where

$$\sigma^2 \leq \lim_{n\to\infty} \left[\mathbb{E}(f_n^2) + 2\sum_{i=1}^{\infty} \mathbb{E}(f_n \hat{T}^i f_n)\right].$$

Furthermore, $\sigma = 0$ if and only if there exists an $\mathcal{F}_{n,0}$-measurable sequence g_n such that $\hat{T} f_n = g_n - \hat{T} g_n$.

The idea of the proof is similar to that of Theorem 3.1. The major difference is that the martingale approximation will result in a martingale difference array. We will use Theorem 3.3 to show that the CLT holds for this array.

Proof of Theorem 3.4. We seek $Y_{n,i} \in L^2(X)$ such that, for fixed n, $Y_{n,i}$ is a martingale difference with respect to $\{\mathcal{F}_{n,i}\}_{i=0}^n$. In particular, that means $Y_{n,i}$ should be $\mathcal{F}_{n,i}$-measurable and $\mathbb{E}(Y_{n,i}|\mathcal{F}_{n,i-1}) = 0$. In addition we wish to find a sequence of functions g_n which are, respectively, measurable with respect to $\mathcal{F}_{n,0}$. We would like g_n to be finite almost everywhere and

$$\hat{T}^i f_n = Y_{n,i} + \hat{T}^{i-1} g_n - \hat{T}^i g_n \tag{3.3}$$

for all $i > 0$. Summing and dividing by $\sqrt{n}$ makes the above expression

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \hat{T}^i f_n = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} Y_{n,i} + \frac{1}{\sqrt{n}} \left[\hat{T}^n g_n - g_n \right].$$

As before, the last term above converges to zero in probability, and thus if we can show the CLT for the martingale difference array then it follows for the sequence of functions f_n.

We can find an expression for g_n by taking conditional expectations of (3.3) with respect to $\mathcal{F}_{n,i-1}$. This yields

$$\mathbb{E}(\hat{T}^i f_n|\mathcal{F}_{n,i-1}) = \mathbb{E}(\hat{T}^{i-1} g_n|\mathcal{F}_{n,i-1}) - \mathbb{E}(\hat{T}^i g_n|\mathcal{F}_{n,i-1}),$$

and by the definition of $\mathcal{F}_{n,i-1}$ we find

$$\mathbb{E}(\hat{T} f_n|\mathcal{F}_{n,0}) = g_n - \mathbb{E}(\hat{T} g_n|\mathcal{F}_{n,0}).$$

Hence we have $g_n = \sum_{i=1}^{\infty} \mathbb{E}(\hat{T}^i f_n|\mathcal{F}_{n,0})$ as a solution to the above equation. By assumption this sum converges absolutely almost surely for each n. The martingale difference array is now defined by (3.3), and by construction we have that $Y_{n,i}$ is $\mathcal{F}_{n,i}$-measurable, $\mathbb{E}(Y_{n,i}|\mathcal{F}_{n,i-1}) = 0$, and $Y_{n,i} = \hat{T}^{i-1} Y_{n,1}$.

In order to apply Theorem 3.3 to the difference array we must show that $\lim_{n\to\infty} \mathbb{E}(Y_{n,1}) < \infty$. Note that the array is stationary and ergodic.

Following the idea of the proof of Theorem 3.1 we define

$$g_n(\lambda) = \sum_{i=1}^{\infty} \lambda^{-(i-1)} \mathbb{E}(\hat{T}^i f_n|\mathcal{F}_{n,0})$$

so that $g_n(\lambda) \in L^2(X)$ and

$$Y_{n,i}(\lambda) = \hat{T}^i f_n + \lambda^{-1} \hat{T}^i g_n(\lambda) - \hat{T}^{i-1} g_n(\lambda)$$

is a martingale difference array approximating $Y_{n,i}$. By definition we also have $\lim_{\lambda\to 1} g_n(\lambda) = g_n$ almost surely, and hence $\lim_{\lambda\to 1} Y_{n,i}(\lambda) = Y_{n,i}$ almost surely as well.

A calculation similar to the one in the previous proof yields

$$\mathbb{E}(Y_{n,1}(\lambda)^2) \leq \mathbb{E}(f_n^2) + 2\sum_{i=1}^{\infty} \mathbb{E}(f_n \hat{T}^i f_n)$$

and as a result,

$$\mathbb{E}(Y_{n,1}^2) \leq \mathbb{E}(f_n^2) + 2\sum_{i=1}^{\infty} \mathbb{E}(f_n \hat{T}^i f_n).$$

Hence,

$$\lim_{n\to\infty} \mathbb{E}(Y_{n,1}^2) \leq \lim_{n\to\infty} \left[\mathbb{E}(f_n^2) + 2\sum_{i=1}^{\infty} \mathbb{E}(f_n \hat{T}^i f_n)\right] < \infty$$

by assumption. Therefore Theorem 3.3 can be applied, and the CLT holds for the martingale difference array and thus for the the sequence of functions f_n. $\square$

CHAPTER 4

Central Limit Theorem for Stationary Processes with Infinite Variance

In this chapter we characterize the diffusion constant σ for stationary precoces generated by dynamical systems, using martingale difference approximations.

4.1 Linear spreading

The main differentiator of our two cases will be the assumptions on conditional expectations, and we name them with respect to this difference. Before stating the theorem we will motivate the main idea. Suppose $\{X_i\}_{i=0}^{\infty}$ is a sequence of random variables with infinite variance. In some situations it is advantageous to consider truncated versions of our random variables,

$$X_{n,i} = X_i \cdot \mathbf{1}_{\{X_i < c_n\}},$$

for some c_n depending on n. In particular, if the second moment of $X_{n,i}$ is bounded in n and if the probability that $X_i > c_n$ is small enough, proving the central limit theorem for X_i may be equivalent to proving it for $X_{n,i}$.

The sequences we study will have a fairly weak form of dependence. This is not uncommon for the types of sequences seen in chaotic dynamical systems. In both cases we study we expect the size of the random variables to become smaller as we take more time steps. We will see these properties on display in the applications of chapter six.

We now present the main result for this section.

Theorem 4.1 (Linear Spreading CLT). *Suppose for $n \in \mathbb{N}$ that $\{(X_{n,i}, \mathcal{F}_{n,i})\}_{i=1}^{n}$ is a supermartingale array of identically distributed random variables on $(\Omega, \mathbb{P})$ such that $\mathcal{F}_{n,i} \subset \mathcal{F}_{n,i+1}$.*

Let $|X_{n,i}| < c_n = \sqrt{n}\ln\ln n$*, let* $\mathbb{E}(X_{n,i}) = \mathcal{O}(1)$*, and* $\mathbb{E}(X_{n,i}^2) = O(\ln n)$*. Lastly, suppose the following hold:*

(a) *Cov*$(X_{n,i}, X_{n,j}) < ke^{-a|i-j|}$ *for some constants* $k, a > 0$*;*

(b) $\mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1}) = \theta X_{n,i-1} + (1-\theta)\mathbb{E}(X_{n,i})$ *for a constant* $\theta \in (0,1)$*;*

(c) $\mathbb{E}(X_{n,i}^2|\mathcal{F}_{n,i-1}) = X_{n,i-1}^2 + \mathcal{O}(X_{n,i-1})$*.*

Then

$$\frac{\sum\limits_{i=1}^{n} X_{n,i} - n\mathbb{E}(X_{n,i})}{\sqrt{n\ln n}} \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 = \dfrac{1+\theta}{1-\theta}\lim\limits_{n\to\infty}\dfrac{\mathbb{E}(X_{n,1}^2)}{\ln n}$*.*

The main idea for the proof is to construct a martingale difference array involving the conditional expectation in (b). We can show that this array satisfies the assumptions of the martingale central limit theorem (Theorem 2.8). From there we will demonstrate that the CLT for the difference array implies a CLT for the original array, completing the proof of the theorem.

Proof. Let $\mathcal{F}_{n,0} = \{\emptyset, \Omega\}$ be the trivial σ-algebra and $Z_{n,i} = X_{n,i} - \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})$ for $n \in \mathbb{N}$ and $1 \le i \le n$. Define $S_{n,0} = 0$ and $S_{n,i} = \sum\limits_{m=1}^{n} Z_{n,m}$, then we claim that for $n \in \mathbb{N}$, $\{(S_{n,i}, \mathcal{F}_{n,i})\}_{i=0}^{n}$ is a martingale array and therefore $\{Z_{n,i}\}_{i=1}^{n}$ is a martingale difference array.

Our σ-algebras are increasing and we already know $X_{n,i}$ to be adapted to $\mathcal{F}_{n,i}$, thus it is clear that $S_{n,i}$ is adapted to $\mathcal{F}_{n,i}$ as well. Note that $\mathbb{E}(Z_{n,i}) = \mathbb{E}(X_{n,i}) - \mathbb{E}(X_{n,i}) = 0$, so that we also have $\mathbb{E}(S_{n,i}) = 0 < \infty$. Lastly, the conditional expectation assumption for martingales is satisfied:

$$\begin{aligned}
\mathbb{E}(S_{n,i}|\mathcal{F}_{n,i-1}) &= \mathbb{E}\left(\sum_{m=1}^{i}[X_{n,m} - \mathbb{E}(X_{n,m}|\mathcal{F}_{n,m-1})]\,\middle|\,\mathcal{F}_{n,i-1}\right) \\
&= \mathbb{E}\left(\sum_{m=1}^{i-1}[X_{n,m} - \mathbb{E}(X_{n,m}|\mathcal{F}_{n,m-1})]\right) \\
&= S_{n,i-1}.
\end{aligned}$$

Therefore $\{(S_{n,i}, \mathcal{F}_{n,i})\}_{i=0}^{n}$ is a martingale array and $\{Z_{n,i}\}_{i=1}^{n}$ is a martingale difference array. Note that if we let $\hat{S}_{n,i} = S_{n,i}/\sqrt{n\ln n}$ and $\hat{Z}_{n,i} = Z_{n,i}/\sqrt{n\ln n}$ then these also define martingale and martingale difference arrays, respectively.

Lemma 4.2. *The martingale array* $\{(\hat{S}_{n,i}, \mathcal{F}_{n,i})\}_{i=0}^{n}$ *satisfies the assumptions of the martingale central limit theorem, and*

$$\hat{S}_{n,n} \xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2)$$

where

$$\hat{\sigma}^2 = (1-\theta^2) \lim_{n\to\infty} \frac{\mathbb{E}(X_{n,1}^2)}{\ln n}.$$

We will address the proof of this lemma later on. For now, our assumptions on the one-step conditional expectations tell us that the convergence in the lemma can be rewritten as

$$(1-\theta)\left[\frac{\sum_{i=1}^{n} X_{n,i} - n\mathbb{E}(X_{n,i})}{\sqrt{n\ln n}}\right] + \frac{\theta X_{n,n} - \theta\mathbb{E}(X_{n,n})}{\sqrt{n\ln n}} \xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2).$$

Due to the size restriction on $X_{n,n}$, it is clear that

$$\frac{\theta X_{n,n} - \theta\mathbb{E}(X_{n,n})}{\sqrt{n\ln n}} \le \mathcal{O}\left(\frac{\ln\ln n}{\sqrt{\ln n}}\right) \to 0$$

so that, in fact, we have

$$(1-\theta)\left[\frac{\sum_{i=1}^{n} X_{n,i} - n\mathbb{E}(X_{n,i})}{\sqrt{n\ln n}}\right] \xrightarrow{d} \mathcal{N}(0, \hat{\sigma}^2).$$

This implies that

$$\frac{\sum_{i=1}^{n} X_{n,i} - n\mathbb{E}(X_{n,i})}{\sqrt{n\ln n}} \to \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = \frac{1-\theta^2}{(1-\theta)^2} \lim_{n\to\infty} \frac{\mathbb{E}(X_{n,1}^2)}{\ln n} = \frac{1+\theta}{1-\theta} \lim_{n\to\infty} \frac{\mathbb{E}(X_{n,1}^2)}{\ln n}.$$

□

It remains to prove Lemma 4.2, that is, to show that the martingale array in question satisfies the assumptions of the martingale central limit theorem. This is largely a matter of calculations and will complete the proof of Theorem 4.1.

Proof of Lemma 4.2. We have already shown that $\{(\hat{S}_{n,i}, \mathcal{F}_{n,i})\}_{i=0}^{n}$ is a martingale array. By definition $S_{n,0} = 0$ and by assumption the σ-algebras are increasing. Recall that the original martingale differences are $Z_{n,i} = X_{n,i} - \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})$. The imposed truncations imply that

$$\max_{1\le i\le n} \|\hat{Z}_{n,i}\|_{L^\infty} \le \mathcal{O}\left(\frac{\ln\ln n}{\sqrt{\ln n}}\right) \to 0.$$

All that remains to be shown is that

$$\sum_{i=1}^{n} \mathbb{E}(\hat{Z}_{n,i}^2 | \mathcal{F}_{n,i-1}) \xrightarrow{L^2} \hat{\sigma}^2.$$

Note that

$$\begin{aligned} Z_{n,i} &= X_{n,i} - \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1}) \\ &= X_{n,i} - \theta X_{n,i-1} - (1-\theta)\mathbb{E}(X_{n,1}) \end{aligned}$$

due to the assumption on the conditional expectation. Squaring this expression yields

$$\begin{aligned} Z_{n,i}^2 &= X_{n,i}^2 + \theta^2 X_{n,i-1}^2 - 2\theta X_{n,i} X_{n,i-1} + (1-\theta)^2 \mathbb{E}(X_{n,1})^2 \\ &\quad - 2(1-\theta)\mathbb{E}(X_{n,1})X_{n,i} + 2\theta(1-\theta)\mathbb{E}(X_{n,1})X_{n,i-1} \end{aligned}$$

Fortunately, taking the conditional expectation with respect to $\mathcal{F}_{n,i-1}$ simplifies this expression greatly. Using the assumptions of Theorem 3.1, we have

$$\begin{aligned} \mathbb{E}(Z_{n,i}^2|\mathcal{F}_{n,i-1}) &= X_{n,i-1}^2 + \theta^2 X_{n,i-1}^2 - 2\theta^2 X_{n,i-1}^2 + \mathcal{O}(X_{n,i-1}) \\ &= (1-\theta^2)X_{n,i-1}^2 + \mathcal{O}(X_{n,i-1}). \end{aligned}$$

These calculations hold true for $2 \leq i \leq n$, but we must do $i = 1$ separately since $\mathcal{F}_{n,0}$ is the trivial σ-algebra. Indeed, $Z_{n,1} = X_{n,1} - \mathbb{E}(X_{n,1})$ and $Z_{n,1}^2 = X_{n,1}^2 + \mathbb{E}(X_{n,1})^2 - 2\mathbb{E}(X_{n,1})X_{n,1}$. Taking the conditional expectation gives us

$$\mathbb{E}(Z_{n,1}^2|\mathcal{F}_{n,0}) = \mathbb{E}(X_{n,1}^2) - \mathbb{E}(X_{n,1})^2.$$

To prove the desired convergence to the constant $\hat{\sigma}^2$ we must show that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^{n} \mathbb{E}(\hat{Z}_{n,i}|\mathcal{F}_{n,i-1}) - \hat{\sigma}^2\right)^2\right] = 0.$$

Based on our previous calculations and the definition of $\hat{Z}_{n,i}$ this can be rewritten as

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{\mathbb{E}(X_{n,1}^2) - \mathbb{E}(X_{n,1})^2}{n \ln n} + \frac{1}{n \ln n} \sum_{i=2}^{n} \left[(1-\theta^2)X_{n,i-1}^2 + \mathcal{O}(X_{n,i-1})\right] - \hat{\sigma}^2\right)^2\right] = 0.$$

Now let us consider the squared term inside the expectation. Expanding it gives the following:

$$
\begin{aligned}
&\left(\frac{\mathbb{E}(X_{n,1}^2)-\mathbb{E}(X_{n,1})^2}{n\ln n}+\frac{1}{n\ln n}\sum_{i=2}^{n}\left[(1-\theta^2)X_{n,i-1}^2+\mathcal{O}(X_{n,i-1})\right]-\hat{\sigma}^2\right)^2 && (4.1)\\
&=\frac{1}{n^2(\ln n)^2}\left[\mathbb{E}(X_{n,1}^2)^2-2\mathbb{E}(X_{n,1}^2)\mathbb{E}(X_{n,1})^2+\mathbb{E}(X_{n,1})^4\right] && (4.2)\\
&\quad+2\left[\frac{\mathbb{E}(X_{n,1}^2)-\mathbb{E}(X_{n,1})^2}{n^2(\ln n)^2}\right]\sum_{i=2}^{n}\left[(1-\theta^2)X_{n,i-1}^2+\mathcal{O}(X_{n,i-1})\right] && (4.3)\\
&\quad-2\hat{\sigma}^2\left[\frac{\mathbb{E}(X_{n,1}^2)-\mathbb{E}(X_{n,1})^2}{n\ln n}\right] && (4.4)\\
&\quad+\frac{1}{n^2(\ln n)^2}\left[\sum_{i=2}^{n}\left[(1-\theta^2)X_{n,i-1}^2+\mathcal{O}(X_{n,i-1})\right]\right]^2 && (4.5)\\
&\quad-\frac{2\hat{\sigma}^2}{n\ln n}\sum_{i=2}^{n}\left[(1-\theta^2)X_{n,i-1}^2+\mathcal{O}(X_{n,i-1})\right]+\hat{\sigma}^4 && (4.6)
\end{aligned}
$$

In what follows we will simplify, take expectations, and take the limit as n approaches infinity for the above, line by line.

In (4.2) note that taking the expectation does not change the expression. In fact, the largest term is $\mathbb{E}(X_{n,1}^2)^2$, which is $\mathcal{O}\left((\ln n)^2\right)$. Therefore these terms converge to zero.

The largest term in line (4.3) is

$$\frac{2\mathbb{E}(X_{n,1}^2)}{n^2(\ln n)^2}\sum_{i=2}^{n}(1-\theta^2)X_{n,i-1}^2,$$

which, upon taking expectations, becomes $2(1-\theta^2)n\mathbb{E}(X_{n,i}^2)^2/(n^2(\ln n)^2)$. This term is of order $\mathcal{O}\left(n(\ln n)^2/n^2(\ln n)^2\right)=\mathcal{O}\left((\ln n)^{-1}\right)$, which again converges to zero in the limit.

It is easy to see that, again, in line (4.4) taking expectations does not change the expression, and this term is $\mathcal{O}\left(n^{-1}\right)\xrightarrow{n\to\infty}0$.

Completing this process for (4.5) is a bit more involved, but will yield a non-zero limit. For now we ignore the $(n \ln n)^{-2}$ factor and focus on expanding the square term.

$$\left[\sum_{i=2}^{n} \left[(1-\theta^2)X_{n,i-1}^2 + \mathcal{O}(X_{n,i-1})\right]\right]^2 \tag{4.7}$$

$$= (1-\theta^2)^2 \sum_{i=2}^{n} X_{n,i-1}^4 \tag{4.8}$$

$$+ 2(1-\theta^2)^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} X_{n,i-1}^2 X_{n,j-1}^2 \tag{4.9}$$

$$+ \sum_{i=2}^{n} \mathcal{O}(X_{n,i-1}^3) + \sum_{i=2}^{n} \mathcal{O}(X_{n,i-1}^2) \tag{4.10}$$

$$+ \sum_{i=2}^{n} \sum_{j=i+1}^{n} \mathcal{O}(X_{n,i-1}^2 X_{n,j}) + \sum_{i=2}^{n} \sum_{j=i+1}^{n} \mathcal{O}(X_{n,i-1} X_{n,j-1}) \tag{4.11}$$

We will tackle this sub-calculation one line at a time as well, taking expectations and then limits.

Upon taking expectations in (4.8) we have the inequality

$$(1-\theta^2)^2 \sum_{i=2}^{n} \mathbb{E}(X_{n,i-1}^4) \leq (1-\theta^2)^2 n c_n^2 \mathbb{E}(X_{n,1}^2) = \mathcal{O}(n^2 \ln n (\ln \ln n)^2).$$

After dividing by $(n \ln n)^2$ this term is of order $\mathcal{O}\left((\ln \ln n)^2 / \ln n\right)$, which limits to zero as n approaches infinity.

The expression in (4.9) is in fact the only one of these terms which does not limit to zero. The expectation of this term satisfies

$$\begin{aligned}
&2(1-\theta^2)^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} \mathbb{E}(X_{n,i-1}^2 X_{n,j-1}^2) \\
&= 2(1-\theta^2)^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} \left[\mathbb{E}(X_{n,i-1}^2)\mathbb{E}(X_{n,j-1}^2) + \mathrm{Cov}(X_{n,i-1}^2, X_{n,j-1}^2)\right] \\
&\leq 2(1-\theta^2)^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} \left[\mathbb{E}(X_{n,1}^2)^2 + c_n^2 \mathrm{Cov}(X_{n,i-1}, X_{n,j-1})\right] \\
&\leq 2(1-\theta^2)^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} \left[\mathbb{E}(X_{n,1}^2)^2 + k c_n^2 e^{-a(j-i)}\right] \\
&= \left((1-\theta^2) n \mathbb{E}(X_{n,1}^2)\right)^2 + \mathcal{O}\left(n^2 (\ln \ln n)^2\right)
\end{aligned}$$

The first term in the above line is order $\mathcal{O}(n^2(\ln n)^2)$ so will limit to a constant after the division, in fact, we will have $\left[(1-\theta^2)\mathbb{E}(X_{n,1}^2)(\ln n)^{-1}\right]^2$, which we will come back to later. The second term in the final line of the inequality above limits to zero.

The largest term in line (4.10) after taking expectations is $\mathcal{O}(n\mathbb{E}(X_{n,1}^3)) \leq \mathcal{O}(nc_n\mathbb{E}(X_{n,1}^2) = \mathcal{O}(nc_n \ln n)$. Dividing by $(n \ln n)^2$ and taking the limit brings this term to zero.

Finally, we can study (4.11) in the same way. Taking expectations, this expression has the following inequality:

$$\begin{aligned}\sum_{i=2}^{n}\sum_{j=i+1}^{n}\mathcal{O}\left(\mathbb{E}(X_{n,i-1}^2X_{n,j-1})\right) &= \sum_{i=2}^{n}\sum_{j=i+1}^{n}\mathcal{O}\left(\mathbb{E}(X_{n,i-1}^2)\mathbb{E}(X_{n,j-1})+\mathrm{Cov}(X_{n,i-1}^2,X_{n,j-1})\right)\\ &\le \sum_{i=2}^{n}\sum_{j=i+1}^{n}\mathcal{O}\left(\mathbb{E}(X_{n,i-1}^2)\mathbb{E}(X_{n,j-1})+c_n\mathrm{Cov}(X_{n,i-1},X_{n,j-1})\right)\\ &\le \mathcal{O}\left(n^2\ln n\right)\end{aligned}$$

Again, dividing by $(n\ln n)^2$ and taking the limit takes this term to zero. This completes the calculations for (4.5). Recall that lines (4.2)-(4.4) were negligibly small. We will now address the final part of the computation.

Taking care of (4.6) is fortunately a straightforward affair. Taking expectations gives

$$-\frac{2\hat{\sigma}^2}{n\ln n}\sum_{i=2}^{n}\left[(1-\theta)^2\mathbb{E}(X_{n,1}^2)+\mathcal{O}(\mathbb{E}(X_{n,1}))\right]+\hat{\sigma}^4.$$

Since $\mathbb{E}(X_{n,1})$ is a constant, after summing we have the term $\mathcal{O}(n\mathbb{E}(X_{n,1})/n\ln n)$ which goes to zero. However, $\mathbb{E}(X_{n,1}^2)$ is order $\mathcal{O}(\ln n)$, so this term does not limit to zero. Therefore, the contribution from (4.6) is

$$-\frac{2\hat{\sigma}^2(1-\theta^2)\mathbb{E}(X_{n,1}^2)}{\ln n}+\hat{\sigma}^4.$$

Putting everything together, we now have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[\left(\sum_{i=1}^{n}\mathbb{E}(\hat{Z}_{n,i}|\mathcal{F}_{n,i-1})-\hat{\sigma}^2\right)^2\right] &= \lim_{n\to\infty}\left[\frac{(1-\theta^2)^2\mathbb{E}(X_{n,1}^2)^2}{(\ln n)^2}-\frac{2\hat{\sigma}^2(1-\theta^2)\mathbb{E}(X_{n,1}^2)}{\ln n}+\hat{\sigma}^4\right]\\ &= \lim_{n\to\infty}\left[\frac{(1-\theta^2)\mathbb{E}(X_{n,1}^2)}{\ln n}-\hat{\sigma}^2\right]^2.\end{aligned}$$

Since $\mathbb{E}(X_{n,1}^2)$ is $\mathcal{O}(\ln n)$ we know that this limit does converge to zero, and in fact this then implies that

$$\hat{\sigma}^2=(1-\theta^2)\lim_{n\to\infty}\frac{\mathbb{E}(X_{n,1}^2)}{\ln n},$$

as desired. This concludes the proof of the Lemma 4.2, and thus the proof of Theorem 4.1.

□

4.2 Algebraic spreading

As was mentioned in the previous section, the main difference between the linear spreading case and the algebraic spreading case will be the assumptions on certain conditional expectations.

Otherwise, aside from a technical point on the range of the random variables, the assumptions for the following case are similar to those we have already seen. As before, one main advantage for this theorem is that it allows us to study some sequences of random variables which have infinite variance and weak dependence. The theorem has applications to certain nonuniformly hyperbolic systems we will see in chapter six.

Theorem 4.3 (Algebraic Spreading CLT). *Suppose for $n \in \mathbb{N}$ that $\{(X_{n,i}, \mathcal{F}_{n,i})\}_{i=1}^n$ is a stationary adapted array of identically distributed natural-valued random variables on $(\Omega, \mathbb{P})$ such that $\mathcal{F}_{n,i} \subset \mathcal{F}_{n,i+1}$. Let $|X_{n,i}| < c_n = \sqrt{n} \ln \ln n$, let $\mathbb{E}(X_{n,i}^{2/\delta}) = \mathcal{O}(1)$ for any $\delta > 1$, and $\mathbb{E}(X_{n,i}^2) = \mathcal{O}(\ln n)$ for some constant c. Lastly, suppose the following hold:*

(a) $Cov(X_{n,i}^{1/\gamma}, X_{n,j}^{1/\gamma}) < ke^{-a|i-j|}$ for all $\gamma \geq 1$ and some $k, a > 0$ depending on γ;

(b) $\mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1}) = \mathcal{O}(X_{n,i-1}^{1/\alpha})$ for some $\alpha > 1$;

(c) $\mathbb{E}(X_{n,i}^2|\mathcal{F}_{n,i-1}) = \beta X_{n,i-1}^2 + o(X_{n,i-1}^2)$ for a constant $\beta > 0$.

Then

$$\frac{\sum_{i=1}^{n} X_{n,i} - n\mathbb{E}(X_{n,1})}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = \beta \lim_{n\to\infty} \dfrac{\mathbb{E}(X_{n,1}^2)}{\ln n}$.

Note that based on the conditional expectations, the size of the random variables tends to decrease at a faster rate, in fact algebraically, compared to the linear spreading case. Due to the nature of this condition we restrict our attention to random variables taking values in the natural numbers. The structure of the proof for this theorem is similar to the previous case; we will construct a martingale difference array in the same way and use the martingale central limit theorem to prove a CLT for the corresponding difference array and, consequently, the original array of random variables.

Proof. As before, let $\mathcal{F}_{n,0} = \{\emptyset, \Omega\}$ be the trivial σ-algebra and $Z_{n,i} = X_{n,i} - \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})$ for $n \in \mathbb{N}$ and $1 \leq i \leq n$. Define $S_{n,0} = 0$ and $S_{n,i} = \displaystyle\sum_{m=1}^{n} Z_{n,m}$. As we saw in the proof of the linear spreading case $\{(S_{n,i}, \mathcal{F}_{n,i})\}_{i=0}^n$ is a martingale array and $\{Z_{n,i}\}_{i=1}^n$ its corresponding martingale difference array. If we let $\hat{S}_{n,i} = S_{n,i}/\sqrt{n \ln n}$ and $\hat{Z}_{n,i} = Z_{n,i}/\sqrt{n \ln n}$ then these too define martingale and martingale difference arrays, respectively.

Lemma 4.4. *The martingale array* $\{(\hat{S}_{n,i}, \mathcal{F}_{n,i})\}_{i=0}^{n}$ *satisfies the assumptions of the martingale central limit theorem, and*

$$\hat{S}_{n,n} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

where

$$\sigma^2 = \beta \lim_{n\to\infty} \frac{\mathbb{E}(X_{n,1}^2)}{\ln n}.$$

We will come back to the proof of this later. Given the assumption on the one-step conditional expectations and the definition of $\mathcal{F}_{n,0}$ we can rewrite the convergence in the lemma as

$$\frac{1}{\sqrt{n \ln n}} \sum_{i=1}^{n} [X_{n,i} - \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})] \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Going one step further, this is equivalent to

$$\frac{\sum_{i=1}^{n} X_{n,i} - n\mathbb{E}(X_{n,1})}{\sqrt{n \ln n}} + \frac{\sum_{i=1}^{n} [\mathbb{E}(X_{n,i}) - \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})]}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Therefore to complete the proof of the theorem it remains only to show that

$$\frac{\sum_{i=1}^{n} [\mathbb{E}(X_{n,i}) - \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})]}{\sqrt{n \ln n}} \xrightarrow{d} 0.$$

In order to do so we will apply Chebyshev's inequality. First we must find the second moment of the sum of the differences in expectation and conditional expectation. The square of this sum is

$$\left[n\mathbb{E}(X_{n,1}) - \sum_{i=1}^{n} \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})\right]^2 = \sum_{i=1}^{n} \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})^2 \tag{4.12}$$

$$+ 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})\mathbb{E}(X_{n,j}|\mathcal{F}_{n,j-1}) \tag{4.13}$$

$$+ n^2\mathbb{E}(X_{n,1}^2) - 2n\mathbb{E}(X_{n,1}) \sum_{i=1}^{n} \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1}) \tag{4.14}$$

We can now find the second moment by taking expectations one line at a time.

For line (4.12) the conditional expectation assumption gives us

$$\sum_{i=1}^{n} \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})^2 = \sum_{i=1}^{n} \mathcal{O}(X_{n,i-1}^{2/\alpha}),$$

and the expectation assumption makes the expectation of (4.12)

$$\sum_{i=1}^{n} \mathcal{O}(\mathbb{E}(X_{n,i-1}^{2/\alpha})) = \mathcal{O}(n).$$

Taking the expectation of (4.13) and utilizing a covariance assumption yields

$$\begin{aligned}
&2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathbb{E}\left(\mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})\mathbb{E}(X_{n,j}\mathcal{F}_{n,j-1})\right)\\
&=2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left[\mathbb{E}(X_{n,1})^2+\mathcal{O}\left(\mathrm{Cov}(X_{n,i-1}^{1/\alpha},X_{n,j-1}^{1/\alpha})\right)\right]\\
&\leq 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left[\mathbb{E}(X_{n,1})^2+\mathcal{O}\left(ke^{-a(j-i)}\right)\right]\\
&=n^2\mathbb{E}(X_{n,1})^2+\mathcal{O}(n).
\end{aligned}$$

Finally, the expectation of the expression (4.14) is equal to $-n^2\mathbb{E}(X_{n,1})^2$. Putting it all together we have

$$\mathbb{E}\left(\left[n\mathbb{E}(X_{n,1})-\sum_{i=1}^{n}\mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})\right]^2\right)=\mathcal{O}(n).$$

Applying Chebyshev's inequality we find that, for all $\epsilon>0$,

$$\mathbb{P}\left(\left|n\mathbb{E}(X_{n,1})-\sum_{i=1}^{n}\mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})\right|\geq\epsilon\sqrt{n\ln n}\right)\leq\frac{\mathcal{O}(n)}{\epsilon^2 n\ln n}\xrightarrow{n\to\infty}0.$$

Convergence in probability implies convergence in distribution, so this gives

$$\frac{\sum_{i=1}^{n}\left[\mathbb{E}(X_{n,i})-\mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})\right]}{\sqrt{n\ln n}}\xrightarrow{d}0$$

and subsequently

$$\frac{\sum_{i=1}^{n}X_{n,i}-n\mathbb{E}(X_{n,1})}{\sqrt{n\ln n}}\xrightarrow{d}\mathcal{N}(0,\sigma^2),$$

as desired. □

It remains only to show that the martingale array in Lemma 4.4 satisfies the assumption of Theorem 2.8, the martingale central limit theorem.

Proof of Lemma 4.4. We have seen that $\{(\hat{S}_{n,i},\mathcal{F}_{n,i})\}_{i=0}^{n}$ is a martingale array. By definition $S_{n,0}=0$ and by assumption we have $\mathcal{F}_{n,i}\subset\mathcal{F}_{n,i+1}$ for $0\leq i\leq n$. Recall that $X_{n,i}<c_n=\sqrt{n}\ln\ln n$ and that $Z_{n,i}=X_{n,i}-\mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1})$. It follows that

$$\max_{1\leq i\leq n}\|\hat{Z}_{n,i}\|_{L^\infty}\leq\mathcal{O}\left(\frac{\ln\ln n}{\sqrt{\ln n}}\right)\to 0.$$

Thus, the only assumption of Theorem 2.8 that remains to be shown is

$$\sum_{i=1}^{n}\mathbb{E}(Z_{n,i}^2|\mathcal{F}_{n,i-1})\xrightarrow{L^2}\sigma^2.$$

From the assumptions of Theorem 4.3 we have

$$\begin{aligned} Z_{n,i} &= X_{n,i} - \mathbb{E}(X_{n,i}|\mathcal{F}_{n,i-1}) \\ &= X_{n,i} - \mathcal{O}(X_{n,i-1}^{1/\alpha}). \end{aligned}$$

Squaring this expression yields

$$Z_{n,i}^2 = X_{n,i}^2 + \mathcal{O}(X_{n,i}X_{n,i-1}^{1/\alpha}) + \mathcal{O}(X_{n,i-1}^{2/\alpha}),$$

and taking the conditional expectation with respect to $\mathcal{F}_{n,i-1}$ gives us

$$\begin{aligned} \mathbb{E}(Z_{n,i}^2|\mathcal{F}_{n,i-1}) &= \beta X_{n,i-1}^2 + o(X_{n,i-1}^2) + \mathcal{O}(X_{n,i-1}^{2/\alpha}) \\ &= \beta X_{n,i-1}^2 + o(X_{n,i-1}^2) \end{aligned}$$

As in the proof for Lemma 4.2, note that the above calculations hold for $2 \le i \le n$. Since $\mathcal{F}_{n,0}$ is the trivial σ-algebra we have the following when $i = 1$:

$$Z_{n,1} = X_{n,1} - \mathbb{E}(X_{n,1}|\mathcal{F}_{n,0}) = X_{n,1} - \mathbb{E}(X_{n,1}).$$

This is the same expression as in the previously mentioned proof, so in fact we again have

$$\mathbb{E}(Z_{n,1}^2|\mathcal{F}_{n,0}) = \mathbb{E}(X_{n,1}^2) - \mathbb{E}(X_{n,1})^2.$$

Now to prove the desired L^2 convergence we must show that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^{n} \mathbb{E}(\hat{Z}_{n,i}|\mathcal{F}_{n,i-1}) - \sigma^2\right)^2\right] = 0.$$

The definition of $\hat{Z}_{n,i}$ and the above calculations allow us to rewrite this limit as

$$\lim_{n\to\infty}\left[\left(\frac{\mathbb{E}(X_{n,1}^2) - \mathbb{E}(X_{n,1})^2}{n\ln n} + \frac{1}{n\ln n}\sum_{i=2}^{n}\left[\beta X_{n,i-1}^2 + o(X_{n,i-1}^2)\right] - \sigma^2\right)^2\right] = 0.$$

In order to prove this statement we begin by expanding the square term inside the expectation.

$$\left(\frac{\mathbb{E}(X_{n,1}^2) - \mathbb{E}(X_{n,1})^2}{n \ln n} + \frac{1}{n \ln n} \sum_{i=2}^{n} \left[\beta X_{n,i-1}^2 + o(X_{n,i-1}^2)\right] - \sigma^2\right)^2 \tag{4.15}$$

$$= \frac{1}{n^2 (\ln n)^2} \left[\mathbb{E}(X_{n,1}^2)^2 - 2\mathbb{E}(X_{n,1}^2)\mathbb{E}(X_{n,1})^2 + \mathbb{E}(X_{n,1})^4\right] \tag{4.16}$$

$$+ 2\left[\frac{\mathbb{E}(X_{n,1}^2) - \mathbb{E}(X_{n,1})^2}{n^2 (\ln n)^2}\right] \sum_{i=2}^{n} \left[\beta X_{n,i-1}^2 + o(X_{n,i-1}^2)\right] \tag{4.17}$$

$$- \sigma^2 \left[\frac{\mathbb{E}(X_{n,1}^2) - \mathbb{E}(X_{n,1})^2}{n \ln n}\right] \tag{4.18}$$

$$+ \frac{1}{n^2 (\ln n)^2} \left[\sum_{i=2}^{n} \left[\beta X_{n,i-1}^2 + o(X_{n,i-1}^2)\right]\right]^2 \tag{4.19}$$

$$- \frac{2\sigma^2}{n \ln n} \sum_{i=2}^{n} \left[\beta X_{n,i-1}^2 + o(X_{n,i-1}^2)\right] + \sigma^4. \tag{4.20}$$

In what follows we will simplify, take expectations, and then take the limit of the above expression line by line.

We begin by noting that lines (4.16) and (4.18) are the same as (4.2) and (4.4) in the proof of Lemma 4.2. Since the assumptions on the first and second moments of the random variables are the same in both the linear and algebraic spreading cases it follows that the expressions in these lines are negligibly small, as was seen previously.

A simple calculation shows that the expectation of line (4.17) is of order $\mathcal{O}(n^{-1})$, which limits to zero as n approaches infinity.

Ignoring the $(n \ln n)^{-2}$ term for now and expanding (4.19) gives

$$\left[\sum_{i=2}^{n} \left[\beta X_{n,i-1}^2 + o(X_{n,i-1}^2)\right]\right]^2 = 2\beta^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} X_{n,i-1}^2 X_{n,j-1}^2 \tag{4.21}$$

$$+ \sum_{i=2}^{n} \mathcal{O}(X_{n,i-1}^4) \tag{4.22}$$

$$+ \sum_{i=2}^{n} \sum_{j=i+1}^{n} o(X_{n,i-1}^2 X_{n,j-1}^2). \tag{4.23}$$

We consider this sub-problem now line by line.

Taking the expectation of (4.21) we have

$$\begin{aligned}
2\beta^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} \mathbb{E}(X_{n,i-1}^2 X_{n,j-1}^2) &= 2\beta^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} \left[\mathbb{E}(X_{n,1}^2)^2 + \mathrm{Cov}(X_{n,i-1}^2, X_{n,j-1}^2)\right] \\
&\leq 2\beta^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} \left[\mathbb{E}(X_{n,1}^2)^2 + c_n^2 \mathrm{Cov}(X_{n,i-1}, X_{n,j-1})\right] \\
&\leq 2\beta^2 \sum_{i=2}^{n} \sum_{j=i+1}^{n} \left[\mathbb{E}(X_{n,1}^2)^2 + k c_n^2 e^{-a(j-i)}\right] \\
&= \left(\beta n \mathbb{E}(X_{n,1}^2)\right)^2 + \mathcal{O}\left(n^2 (\ln\ln n)^2\right)
\end{aligned}$$

We see that the first term above, $\left(\beta n \mathbb{E}(X_{n,1}^2)\right)^2$, is of $\mathcal{O}\left(n^2(\ln n)^2\right)$, so it does not limit to zero after we divide. The second term on the other hand is negligibly small.

When considering line (4.22) we see that

$$\sum_{i=2}^{n} \mathcal{O}(\mathbb{E}(X_{n,i-1}^4)) \leq \sum_{i=2}^{n} \mathcal{O}(c_n^2 \mathbb{E}(X_{n,i-1}^2)) = \mathcal{O}\left(n^2 (\ln\ln n)^2 \ln n\right),$$

which upon dividing by $n^2(\ln n)^2$ is of order $\mathcal{O}\left((\ln\ln n)^2/\ln n\right)$, which also limits to zero as n approaches infinity.

Lastly, we note that line (4.23) is of small order of the expression in (4.21), so that after taking expectations it has order $o\left(n^2(\ln n)^2\right)$. Again this term will go to zero. Therefore, upon taking the expected value, the only non-negligible term in (4.19) is $\left(\beta n \mathbb{E}(X_{n,1}^2)\right)^2$.

We now return to (4.20). The expectation of this expression is

$$-\frac{2\sigma^2}{n\ln n}(n-1)\left[\beta\mathbb{E}(X_{n,i-1}^2) + o\left(\mathbb{E}(X_{n,i-1}^2)\right)\right] + \sigma^4 = -\frac{2\sigma^2}{n\ln n}(n-1)\left[\beta\mathbb{E}(X_{n,i-1}^2) + o\left(\ln n\right)\right] + \sigma^4.$$

The non-negligible terms as n approaches infinity are $-\dfrac{2\beta\sigma^2\mathbb{E}(X_{n,1}^2)}{\ln n} + \sigma^4$.

Compiling these calculations, we can now write

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{i=1}^{n} \mathbb{E}(\hat{Z}_{n,i}|\mathcal{F}_{n,i-1}) - \sigma^2\right)^2\right] &= \lim_{n\to\infty}\left[\left(\frac{\beta\mathbb{E}(X_{n,1}^2)}{\ln n}\right)^2 - \frac{2\beta\sigma^2\mathbb{E}(X_{n,1}^2)}{\ln n} + \sigma^4\right] \\
&= \lim_{n\to\infty}\left[\frac{\beta\mathbb{E}(X_{n,1}^2)}{\ln n} - \sigma^2\right]^2
\end{aligned}$$

It is now easy to see that for $\sigma^2 = \lim\limits_{n\to\infty} \dfrac{\beta\mathbb{E}(X_{n,1}^2)}{\ln n}$ this limit does indeed converge to zero, proving that

$$\sum_{i=1}^{n} \mathbb{E}(\hat{Z}_{n,i}|\mathcal{F}_{n,i-1}) \xrightarrow{L^2} \sigma^2,$$

and therefore that the martingale array $\{(\hat{S}_{n,i}, \mathcal{F}_{n,i})\}_{i=0}^{n}$ does satisfy the assumptions of Theorem 2.8. This concludes the proof of Lemma 4.4 and, as a result, the proof of Theorem 4.3. □

CHAPTER 5

Central Limit Theorem for Nonuniformly Hyperbolic Systems

The following discussion is largely inspired by Wang, Zhang, and Zhang's paper [87]. In fact, the assumptions on the systems and related theories can be found in their work, aside from the linear and square spreading assumptions and the theory in Section 5.4, which requires the application of our Theorems 4.1 and 4.3.

5.1 Nonuniformly hyperbolic systems

Let Ω be a compact 2-dimensional Riemannian manifold, and $\hat{S}_0 \subset \Omega$ a compact set consisting of at most countably many smooth compact curves. Then $\mathcal{M} = \Omega \backslash \hat{S}_0$ is made of at most countably many connected open components. Assume that $T\colon \mathcal{M} \to \mathcal{M}$ is a local $C^{1+\gamma}$ diffeomorphism on each connected component of $\mathcal{M}$ with $\gamma \in (0,1)$. We extend T by continuity to the boundary of each open region in $\mathcal{M}$ and call $\hat{S}_0 \cup \mathcal{F}^{-1}\hat{S}_0$ the singular set. We will assume that T preserves a Borel measure μ and the system (T, μ) is mixing and *nonuniformly hyperbolic*. More precisely, either there exists a null set of points with zero Lyapunov exponents; or the stable/unstable cones are not strictly invariant at every iteration. Let $d(\cdot, \cdot)$ be the distance function on $\Omega \times \Omega$ induced by the Riemannian metric in Ω. For any smooth curve W in Ω we denote by $|W|$ its Lebesgue length. For any measurable set $A \subset \mathcal{M}$ we denote by μ_A the conditional measure on A induced by μ.

We wish to study the limiting behavior of Birkhoff sums

$$\mathcal{S}_n f = f + f \circ T + \cdots + f \circ T^{n-1} \tag{5.1}$$

for Hölder continuous functions on $\mathcal{M}$. As usual we consider centered sums, i.e., $\mathcal{S}_n f - n\mu(f) = \mathcal{S}_n(f - \mu(f))$, so we will always assume that f is centralized, that is, $\mu(f) = 0$; otherwise we replace f with $f - \mu(f)$.

Our main goal is to establish the CLT:

Conjecture 5.1. *Suppose a nonuniformly hyperbolic system satisfies the assumptions* **(H1)** *-* **(H3)** *of Section 5.3, and let $f \in L^\infty(\mathcal{M}, \mu)$ be a bounded, piecewise Hölder continuous function with exponent $\alpha \in (0, 1]$, with continuity contained in the singularity of T^{n_0}, for some $n_0 \geq 1$. Then as $n \to \infty$,*

$$\frac{\mathcal{S}_n f}{\sigma\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(0, 1). \tag{5.2}$$

Since the map T is ergodic, the limit law (5.2) holds if we replace μ with any measure that is absolutely continuous with respect to μ.

In studying statistical properties of $(T, \mathcal{M})$, one first needs to study the reduced map by localizing spots in the phase space where expansion (contraction) of tangent vectors slows down. Assume that there exists $M \subset \mathcal{M}$ such that the return map $F : M \to M$ (which avoids 'bad' portions of the phase space $\mathcal{M}$) is uniformly hyperbolic and enjoys exponential decay of correlations. One can easily check that it preserves the measure μ_M obtained by conditioning μ on M. For any $x \in M$ we call $\mathcal{R}(x) = \min\{n \geq 1 : T^n(x) \in M\}$ the return time function and the return map $F\colon M \to M$ is defined by

$$F(x) = T^{\mathcal{R}(x)}(x), \quad \forall x \in M. \tag{5.3}$$

5.2 Assumptions on the reduced map

Let $d(\cdot\,,\cdot)$ be the distance between two points on M induced by the Riemannian metric on M, and m be the corresponding Lebesgue measure on M. Let $|W|$ be the Lebesgue length of W. In this subsection, we list and briefly explain the assumptions on the reduced map.

(h.1) Hyperbolicity of F. There exist two families of cones $C_x^{s/u}$ in the tangent space T_xM, which are continuous on $x \in M$, with the following properties:

(i) The angle between C_x^u and C_x^s is uniformly bounded away from zero, $x \in M$.

(ii) $DF(C_x^u) \subset C_{Fx}^u$ and $DF(C_x^s) \supset C_{Fx}^s$, whenever DF exists.

(iii) There exists a constant $\Lambda > 1$, such that

$$\|D_x F(v)\| \geq \Lambda \|v\|, \ \forall \ v \in C_x^u \quad \text{and} \quad \|D_x F^{-1}(v)\| \geq \Lambda \|v\|, \quad \forall \ v \in C_x^s. \tag{5.4}$$

Remark. A smooth curve W is said to be an *unstable* or a *u-curve*, if at every point $x \in W$ the tangent line $T_x W$ belongs in the unstable cone C_x^u. If $F^{-n}(W)$ are unstable curves for all $n \geq 0$, then W is called an *unstable manifold*, and we denote it W^u. Stable (or s-)curves and stable manifolds W^s are defined in exactly the same way.

(h.2) Singularities.

Let $\mathcal{S}_{\pm 1} = \mathcal{S}_0 \cup F^{\mp 1}\mathcal{S}_0$ be the singular set of $F^{\pm 1}$. We assume that:

(i) The singular curves in $\mathcal{S}_0$ terminate on each other, or on the boundaries of M; the angles between tangent vectors of $\mathcal{S}_{\pm 1}$ and $\mathcal{C}^{s/u}$ have a positive lower bound; and $\mathcal{S}_1$ consists of at most countably many C^2 curves;

(ii) There exist constants $\tau \in (0, 1]$ and $C > 0$ such that for any $x \in M \setminus \mathcal{S}_1$

$$\|D_x F\| \leq C \, d(x, \mathcal{S}_1)^{-\tau}, \tag{5.5}$$

and for any $\varepsilon > 0$,

$$\mu_M(\{x \in M \ : \ d(x, \mathcal{S}_1) \leq \varepsilon\}) \leq c\varepsilon^\tau. \tag{5.6}$$

(iii) **One-step expansion.**

$$\liminf_{\delta_0 \to 0} \sup_{W \, : \, |W| < \delta_0} \sum_{V_\beta \subset FW \setminus \mathcal{S}_{-1}} \left(\frac{|W|}{|V_\beta|} \right)^\tau \cdot \frac{|F^{-1} V_\beta|}{|W|} < 1, \tag{5.7}$$

where the supremum is taken over all unstable curve $W \subset M$.

Remark: It follows from the Borel-Cantelli Lemma and (ii) that, for almost all $x \in M$, the stable/unstable manifolds $W^{s/u}(x)$ exist, and

$$m(\{x \in M \ : \ |W^{s,u}(x)| \leq \varepsilon\}) \leq C\varepsilon^\tau, \tag{5.8}$$

To guarantee the distortion bounds, one can add countably many connected fake singular lines in $\mathcal{S}_0$. In addition we add a finite number of grid lines in M, to guarantee that the lengths of unstable/stable curves are uniformly bounded by a small constant c_M. i.e. for any unstable/stable curves W, we have

$$|W| < c_M. \tag{5.9}$$

Let $\mathcal{S}_{\pm n} = \cup_{m=0}^{n} F^{\mp m}\mathcal{S}_0$ and $\mathcal{S}_{\pm\infty} = \cup_{m\geq 0}\mathcal{S}_{\pm m}$. We use $\mathcal{W}^s$ to denote the collection of all maximal stable manifolds in $M \setminus \mathcal{S}_{-\infty}$, and $\mathcal{W}^u$ to denote the collection of all maximal unstable manifolds in $M \setminus \mathcal{S}_{\infty}$. Note that for every $W \in \mathcal{W}^{s/u}$, the end points of W are on the singular curves $\mathcal{S}_{\pm\infty}$. In addition, (ii) implies that both stable and unstable manifolds form angles $\geq \alpha_0 > 0$, with singular curves in $\mathcal{S}_{\pm 1}$ at intersection points. Moreover, the one-step expansion estimate (iii) guarantees the uniform growth (on average) in each iteration.

The assumption (**h.3**) in the following is about the regularity of stable and unstable manifolds of F. We must point out that the assumption (**h.3**) can be greatly simplified for smooth hyperbolic systems, such as Anosov diffeomorphisms. This is because, in these cases, $\mathcal{S}_0$ only consists of a finite number of fake singular curves to guarantee (5.9), thus (i)-(iii) automatically hold.

(h.3) Regularity of stable/unstable manifolds.

We assume that there is a class of F-invariant stable/unstable manifolds such that every element W is regular in the following sense:

(i) **Bounded curvature.** There exists a small constant $k_0 > 0$, such that at any point $x \in W$, the curvature of W is bounded from above by k_0.

(ii) **Distortion bounds of F.** There exist constants $C_{\mathbf{J}} > 1$ and $\gamma \in (0,1)$, such that for each unstable manifold $W \subset M$, and each pair of points $x,\ y \in W$,

$$\left| \ln J_W F^{-1}(x) - \ln J_W F^{-1}(y) \right| \leq C_{\mathbf{J}}\, d_W(x,y)^{\gamma}, \tag{5.10}$$

where $J_W F^{-1}$ is the Jacobian of F^{-1} along W.

(iii) **Absolute continuity.** For each pair of regular unstable manifolds W^1 and W^2, which are close enough, we define

$$W_*^i := \{x \in W^i \colon W^s(x) \cap W^{3-i} \neq \emptyset\},$$

for $i = 1, 2$. The unstable holonomy map $\mathbf{h} : W_*^1 \to W_*^2$ along stable manifolds is absolutely continuous with uniformly bounded Jacobian $J_{W_*^1}\mathbf{h}$. Furthermore, for each $x,\ y \in W_*^1$,

$$| \ln J_{W_*^1}\mathbf{h}(x) - \ln J_{W_*^1}\mathbf{h}(y)| \leq C_{\mathbf{J}} d_W(x,y)^{\gamma}. \tag{5.11}$$

Remark: Formula (5.10) and the property of uniform hyperbolicity implies that, for each unstable manifold $W \subset M$, and each $k \geq 1$,

$$\left|\ln J_W F^{-1}(F^{-k}x) - \ln J_W F^{-1}(F^{-k}y)\right| \leq C_{\mathbf{J}} d_W(F^{-k}x, F^{-k}y) \leq c_M C_{\mathbf{J}} \Lambda^{-k\gamma}, \quad \forall x, y \in W. \tag{5.12}$$

Accordingly, for each $n \in \mathbb{N}$ and unstable manifold W such that $F^n W$ is smooth, the expansion factor is almost constant on W. In applications, in order to make sure that a given unstable manifold W is regular, one needs to add finitely many grids on M to make sure c_M is small. From now on, unless otherwise specified, all the stable/unstable manifolds in this chapter are the regular stable/unstable manifolds.

According to results in [43, 44] that it follows from assumption **(h1)-(h3)** that the system F preserves an SRB measure μ_M. Moreover, we assume that F^n is ergodic for any $n \geq 1$, which implies that μ_M is a mixing SRB measure.

For each $p \in (1, \infty]$, let $\mathcal{H}_p^+(\gamma)$ be the set of all real-valued functions $f \in L^p(M, \mu_M)$, such that f is Hölder continuous on each connected component of $M \setminus \mathcal{S}_n$, for some $n \geq 0$. We define

$$\|f\|_{p,\gamma} := \|f\|_p + \|f\|_{C^\gamma},$$

where $\|f\|_p = \mu_M(|f|^p)^{\frac{1}{p}}$, and

$$\|f\|_{C^\gamma} := \sup_{k\geq 1} \sup_{x,y \in M_k} \frac{|f(x) - f(y)|}{d(x,y)^\gamma} < \infty.$$

Here $\cup_{k\geq 1} M_k$ is the collection of all disjoint, connected components of $M \setminus \mathcal{S}_n$. Similarly, we define $\mathcal{H}_p^-(\gamma)$ as the set of all real-valued functions $g \in L^p(M, \mu_M)$, such that g is Hölder continuous on each connected component of $M \setminus \mathcal{S}_{-m}$, for some $m \geq 0$. Moreover, if we are going to consider the autocorrelations of certain observables we need to require that these observables belong to the space $\mathcal{H}_p(\gamma) := \mathcal{H}_p^+(\gamma) \cap \mathcal{H}_p^-(\gamma)$, which consists of piecewise Hölder functions with exponent γ and with discontinuities contained in $\mathcal{S}_1 \cup \mathcal{S}_{-1}$.

Next we consider the decay rate of the observables $f \in \mathcal{H}_p^-(\gamma)$ and $g \in \mathcal{H}_p^+(\gamma)$, for $p > 1$. It was proved in [87] that the correlation $\mathcal{C}_n(f, g, F)$ decays exponentially as $n \to \infty$ under the above assumptions.

Lemma 5.2. *Let $p \in (1, \infty]$. Assume (F, μ_M, M) satisfies assumption **(h1-h3)**. Then there exists $\vartheta = \vartheta(p) \in (0,1)$ such that for all $f \in \mathcal{H}_p^-(\gamma)$, and $g \in \mathcal{H}_p^+(\gamma)$, there exists $N = N(f,g) \geq 1$, such that for any $n > N$,*

$$|\mathcal{C}_n(f, g, F)| \leq C \|f\|_{p,\gamma} \|g\|_{p,\gamma} \vartheta^n, \tag{5.13}$$

where $C > 0$ is a constant.

5.3 Assumptions on the nonuniformly hyperbolic systems

(H1) The reduced map $F\colon M \to M$ enjoys exponential decay of correlations.

(H2) The distribution of the return time function $\mathcal{R}\colon \mathcal{M} \to \mathbb{N}$ satisfies:

$$\mu(M_n) = \mathcal{O}(n^{-3}), \tag{5.14}$$

where $M_n = \{x \in M \colon \mathcal{R}(x) = n\}$. Moreover, we assume

$$\mathcal{M} = \cup_{k\geq 0} \cup_{m=1}^{k+1} T^k M_m. \tag{5.15}$$

(H3) Assume there exists $\alpha \in (0, 1]$ such that

(**i**): $\mathbb{E}(\mathcal{R}(F^m(x))|\mathcal{R}(F^{m-1}(x)) = n) = \mathcal{O}(n^\alpha)\mathbf{1}_{M_n}$.

(**ii**): $\mathbb{E}(\mathcal{R}^2(F^m(x))|\mathcal{R}(F^{m-1}(x)) = n) = \mathcal{O}(n^2)\mathbf{1}_{M_n}$.

We also define the induced function $\tilde{f} = f + f \circ T + \cdots + f \circ T^{\mathcal{R}-1}$ and denote by $S_n\tilde{f}$ its Birkhoff sums:

$$S_n\tilde{f} = \tilde{f} + \tilde{f} \circ F + \cdots + \tilde{f} \circ F^{n-1} \tag{5.16}$$

It follows from the Kac's formula that $\mu_M(\mathcal{R}) = 1/\mu(M)$ and $\mu_M(\tilde{f}) = \mu(f)/\mu(M)$. Since we always assume $\mu(f) = 0$ we also have $\mu_M(\tilde{f}) = 0$. If the original function f is continuous then the discontinuity lines of $\tilde{f}$ will coincide with those of the map F.

Conjecture 5.3. *(CLT for the induced map). Let the dynamical system and $f : M \to \mathbb{R}$ satisfy the assumptions of Theorem 5.1 and $\tilde{f}$ be the induced function on M defined above. Then*

$$\frac{S_n\tilde{f}}{A_n} \Rightarrow N(0, 1) \tag{5.17}$$

in distribution, and

$$A_n := \tilde{\sigma}\sqrt{n \ln n}, \tag{5.18}$$

where $\tilde{\sigma}^2 = \sigma^2/\mu(M)$.

Note that the function $\mathcal{R} - \mu_M(\mathcal{R})$ also satisfies the above limit theorem, indeed, $\mathcal{R}$ is induced by the constant function $f(x) = 1$. In addition, the ergodicity of T implies that of F, hence the limit distribution of the left hand side of (5.17) is the same with respect to any measure that is absolutely continuous with respect to μ_M.

Proposition 5.4. *Conjecture 5.1 follows from Conjecture 5.3.*

Proof. Define a measure ν on M such that $\frac{d\nu}{d\mu_M}(x) = m\mu(M)$, for any $x \in M_m$. Clearly, (5.17) holds with respect to the measure ν.

Given $n > 1$ we fix $n' = [n/\mu_M(\mathcal{R})]$. For every $x \in M$, let

$$n'' = \#\{m \leq n \, : \, T^m x \in M\}$$

Then n'' denotes the number of returns to M of the forward trajectory of x within n iterations. Moreover it implies that $S_{n''}\mathcal{R} \leq n \leq S_{n''+1}\mathcal{R}$. Apply (5.17) to $\mathcal{R}$, we know that for $n' \to \infty$,

$$\frac{S_{n''}\mathcal{R} - n''\mu_M(\mathcal{R})}{A_{n''}} \Rightarrow \hat{Y},$$

where $\hat{Y}$ has a normal distribution. This implies that $\frac{(n'-n'')\mu_M(\mathcal{R})}{A_{n''}} \Rightarrow \hat{Y}$ as $n \to \infty$. Thus $\{\frac{(n'-n'')\mu_M(\mathcal{R})}{A_{n''}}\}$ is tight, which implies that for any $\varepsilon > 0$, there exists $C > 0$ such that

$$\nu(|n' - n''| \leq CA_{n''}) \geq 1 - \varepsilon$$

Note that for n large, $n' > n''$, thus

$$I_{1,n} := \frac{S_{n'}\tilde{f} - S_{n''}\tilde{f}}{A_n} = \frac{S_{n'-n''}\tilde{f}}{A_n} = \frac{S_{n'-n''}\tilde{f}}{A_{n'-n''}} \cdot \frac{A_{n'-n''}}{A_n} \Rightarrow 0$$

Next, we have

$$\mathcal{S}_n f - S_{n''}\tilde{f} \leq \|f\|_\infty \left(n - S_{n''}\mathcal{R}\right).$$

Note that $S_{n''}\mathcal{R}(x) \leq n$, thus $n - S_{n''}\mathcal{R}(x) = k$ implies that $T^n(x) \in T^k(M_m)$, for some $m > k$. Then

$$\mu_M(n - S_{n''}\mathcal{R}(x) = k) \leq \sum_{m \geq k} \mu_M(M_m) \leq Ck^{-2}.$$

As a result the following process also converges to zero in probability:

$$I_{2,n} := \frac{\mathcal{S}_n f - S_{n''}\tilde{f}}{A_n} \leq \|f\|_\infty \frac{n - S_{n''}\mathcal{R}}{A_n} \Rightarrow 0 \tag{5.19}$$

Combining the above facts as well as Theorem 5.3, we have shown the limit law (5.2) holds with respect to the measure ν,

$$\frac{\mathcal{S}_n f}{A_n} = \frac{S_{n'}\tilde{f}}{A_n} + \frac{S_{n''}\tilde{f} - S_{n'}\tilde{f}}{A_n} + \frac{\mathcal{S}_n f - S_{n''}\tilde{f}}{A_n} \Rightarrow Y \tag{5.20}$$

where $f : M \to \mathbb{R}$ is a Holder observable.

Next we consider the measure μ. Since $\mathcal{M}$ can be built into a tower based on M with height function $\mathcal{R}$, (M, ν) can be viewed as isometric to the space $(\mathcal{M}, \mu)$. For any $x \in \mathcal{M}$, let $\pi : \mathcal{M} \to M$ be the projection onto the base along trajectories. For any H older function $f : \mathcal{M} \to \mathbb{R}$, if $x \in \mathcal{M} \setminus M$ and $\min\{k \geq 1 : T^k x \in M\} = i$ then x belongs to the i-th level of the tower. For any $n > 1$ and $x \in \mathcal{M}$, let n'' be the number of returns to the base M within n iterations along the trajectory of x. Then according to (5.19), the following sequence converges to zero in probability:

$$\frac{\mathcal{S}_n f(x) - \mathcal{S}_n f \circ \pi(x)}{A_n} \leq \|f\|_\infty \frac{n - S_{n''}\mathcal{R}(\pi(x))}{A_n} \Rightarrow 0.$$

Then

$$\frac{\mathcal{S}_n f}{A_n} = \frac{\mathcal{S}_n f \circ \pi}{A_n} + \frac{\mathcal{S}_n f - \ S_n f \circ \pi}{A_n} \Rightarrow Y.$$

This implies Theorem 5.1. □

Thus it is enough for us to prove Theorem 5.3 from now on.

5.4 CLT for linear and square spreading systems

Based on our knowledge of nonuniformly hyperbolic systems with slow decay of correlations, we mainly consider two types of systems.

Linear spreading. Assume that for any $x \in M_m$, $Fx \in M_n$, with $n \in \mathcal{B}_m \colon = [m/\beta + c_1, \beta m + c_2]$ for some constants $c_1, c_2 > 0$. Assume the transition probability from M_m to M_n satisfies

$$p_{m,n} \colon = \mu_M(M_n | F(M_m)) = \frac{c_0 m}{n^2} + O(m^{-2}) \tag{5.21}$$

where $c_0 = \left[\beta - \beta^{-1}\right]^{-1}$ is the normalizing constant.

Let us show that these systems satisfy (**H3**).

Lemma 5.5. *For Type I systems, (**H3**) is satisfied.*

Proof. Since $\{\mathcal{R} \circ F^i\}$ is stationary, it is enough to calculate the initial one-step conditional expectations. We have

$$\begin{aligned} \mathbb{E}(\mathcal{R} \circ F(x) | \mathcal{R}(x) = m) &= \sum_{n \in \mathcal{B}_m} \left[\frac{c_0 m}{n} + \mathcal{O}(nm^{-2})\right] \mathbf{1}_{M_m} & (5.22)\\ &= \left[2 \ln \beta \left[\beta - \beta^{-1}\right]^{-1} m + \mathcal{O}(1)\right] \mathbf{1}_{M_m} & (5.23) \end{aligned}$$

and

$$\mathbb{E}(\mathcal{R}^2 \circ F(x)|\mathcal{R}(x) = m) = \sum_{n \in \mathcal{B}_m} \left[c_0 m + \mathcal{O}(n^2 m^{-2})\right] \mathbf{1}_{M_m} \tag{5.24}$$

$$= \left[m^2 + \mathcal{O}(m)\right] \mathbf{1}_{M_m} \tag{5.25}$$

□

Algebraic spreading. Assume that for any $x \in M_m$, $Fx \in M_n$, with $n \in \mathcal{B}_m \colon = [c_1 \sqrt[a]{m}, c_2 m^a]$, for some constants $c_1 > 0, c_2 > 0$, and $a > 1$. Assume the transition probability from M_m to M_n satisfies

$$p_{n,m} = \mathcal{O}\left(\frac{m^d}{n^b}\right) \tag{5.26}$$

where $b - 2 > a(d-1)$ and we have two possible cases:

(i) $2 < b < 3$ and $d + a(3-b) \leq 2$, or

(ii) $b \geq 3$ and $1 \leq d < 2$.

Note in particular that if $d \in (0,1)$, then $b - 2 > a(d-1)$ is automatically satisfied in case (i). Let us show that these systems satisfy (**H3**).

Lemma 5.6. *For Type II systems, (**H3**) is satisfied.*

Proof. Since $\{\mathcal{R} \circ F^i\}$ is stationary, it is enough to calculate the initial one-step conditional expectations. Let us first consider case (i). We have

$$\mathbb{E}(\mathcal{R} \circ F(x)|\mathcal{R}(x) = m) = \sum_{n \in \mathcal{B}_m} \mathcal{O}\left(\frac{m^d}{n^{b-1}}\right) \mathbf{1}_{M_m} = \mathcal{O}(m^{d-(b-2)/a})\mathbf{1}_{M_m} \tag{5.27}$$

and

$$\mathbb{E}(\mathcal{R}^2 \circ F(x)|\mathcal{R}(x) = m) = \sum_{n \in \mathcal{B}_m} \mathcal{O}\left(\frac{m^d}{n^{b-2}}\right) \mathbf{1}_{M_m} = \mathcal{O}(m^{d+a(3-b)})\mathbf{1}_{M_m} \leq \mathcal{O}(m^2)\mathbf{1}_{M_m}. \tag{5.28}$$

In the case of (ii), we find that

$$\mathbb{E}(\mathcal{R} \circ F(x)|\mathcal{R}(x) = m) = \sum_{n \in \mathcal{B}_m} \mathcal{O}\left(\frac{m^d}{n^{b-1}}\right) \mathbf{1}_{M_m} = \mathcal{O}(m^{d-(b-2)/a})\mathbf{1}_{M_m} \tag{5.29}$$

and

$$\mathbb{E}(\mathcal{R}^2 \circ F(x)|\mathcal{R}(x) = m) = \sum_{n \in \mathcal{B}_m} \mathcal{O}\left(\frac{m^d}{n^{b-2}}\right) \mathbf{1}_{M_m} = \mathcal{O}(m^{d-(b-3)/a}) \mathbf{1}_{M_m} \leq \mathcal{O}(m^2) \mathbf{1}_{M_m}. \quad (5.30)$$

□

Given the above lemmas we can prove a special case of Conjecture 5.3.

Theorem 5.7. *Suppose a nonuniformly hyperbolic system satisfies **(H1)** and **(H2)** and exhibits either the linear or square spreading property. Then for $\tilde{f} = \mathcal{R} - \mathbb{E}(\mathcal{R})$,*

$$\frac{S_n \tilde{f}}{A_n} \Rightarrow N(0,1) \quad (5.31)$$

in distribution, and

$$A_n := \sigma \sqrt{n \ln n} \quad (5.32)$$

Proof. The main idea here is to use the truncation and theorems introduced in Chapter 4. Let $X_{n,i} = (\mathcal{R} \circ F^i)\mathbf{1}_{\{\mathcal{R} \circ F^i < c_n\}}$, where $c_n = \sqrt{n \ln n}$. Then, due to (**H2**), we have

$$\mathbb{E}(X_{n,i}^2) = \sum_{m=1}^{c_n} m \mu_M(M_m) = \mathcal{O}(\ln n) \quad (5.33)$$

and for any $\varepsilon > 0$, $\mathbb{E}(X_{n,i}^{2-\varepsilon})$ is a constant. Furthermore, by assumption we have exponential decay of correlations, so condition (a) of Theorems 4.1 and 4.3 is satisfied.

We define the σ-algebras $\mathcal{F}_{n,i} = \sigma(\mathcal{R}, \mathcal{R} \circ F, \ldots, \mathcal{R} \circ F^i)$ and let $\mathcal{F}_{n,-1} = \{\emptyset, M\}$. These are increasing, as desired, and it is easy to see that $\{X_{n,i}\}$ is adapted to $\{\mathcal{F}_{n,i}\}$. Furthermore, due to our calculations in Lemmas 5.5 and 5.6 it follows that systems with linear spreading satisfy assumptions (b) and (c) of Theorem 4.1, and systems with algebraic spreading satisfy assumptions (b) and (c) of Theorem 4.3. Therefore those results can be applied to these systems and the central limit theorem holds for the truncated return time function. Also note that for the linear spreading case we have calculated that

$$\theta = 2 \ln \beta \left[\beta - \beta^{-1}\right]^{-1}, \quad (5.34)$$

so that

$$\sigma^2 = c_0 \frac{\beta - \beta^{-1} + 2 \ln \beta}{\beta - \beta^{-1} - 2 \ln \beta}, \quad (5.35)$$

where $c_0 = \lim_{n \to \infty} \dfrac{\mathbb{E}(X_{n,1}^2)}{\ln n}$ and β is the linear spreading factor.

In order to complete the proof it is necessary to show that

$$\frac{1}{\sqrt{n \ln n}} \sum_{i=0}^{n-1} \left[(\mathcal{R} \circ F^i) \mathbf{1}_{\{\mathcal{R} \circ F^i > c_n\}} - \mathbb{E}((\mathcal{R} \circ F^i) \mathbf{1}_{\{\mathcal{R} \circ F^i > c_n\}}) \right] \xrightarrow{d} 0. \tag{5.36}$$

This follows from assumption (**H2**), which implies that

$$\mu_M(\exists i \leq n : F^i(x) \in \bigcup_{m=c_n}^{\infty} M_m) = \mathcal{O}\left((\ln \ln n)^{-2}\right) \to 0. \tag{5.37}$$

In other words, the probability that $\mathcal{R} \circ F^i(x) > c_n$ for any $0 \leq i \leq n$ is negligibly small. Thus, the CLT for the truncated return time funtion implies the CLT for the return time function, completing the proof of Theorem 5.7. $\square$

CHAPTER 6

Applications to Certain Dynamical Systems

6.1 Introduction to billiards

In this section we give some basic definitions and facts concerning billiards. The notation and presentation given here is inspired by that found in Chernov and Markarian's text [41].

Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded open connected domain whose boundary is a finite union of C^3 compact curves:

$$\partial\mathcal{D} = \Gamma_1 \cup \cdots \cup \Gamma_l.$$

Then $\mathcal{D}$ is called a *billiard table* and $\Gamma_1, \ldots, \Gamma_l$ are its *walls*. To generate dynamics we let a point-like particle move inside the billiard table with unit velocity. Upon colliding with a wall the particle bounces instantaneously such that its angle of incidence is equal to its angle of reflection. These dynamics are referred to as the *billiard flow* on $\mathcal{D}$. The billiard flow induces a first return map T to $\partial\mathcal{D}$, often referred to as the *billiard map*. We will be studying the discrete-time dynamics of the billiard map and its associated statistical properties.

Assume that $\partial\mathcal{D}$ has a counterclockwise orientation. By construction we have

$$T : \partial\mathcal{D} \times [-\pi/2, \pi/2] \to \partial\mathcal{D} \times [-\pi/2, \pi/2],$$

and the coordinates of the billiard map are given by (r, φ) where r is an arc length parameter on the boundary of the billiard table and φ is the angle of reflection relative to the normal direction. It is known that the billiard map preserves a probability measure on the collision space $\mathcal{M} = \partial\mathcal{D} \times [-\pi/2, \pi/2]$, given by

$$d\mu = (2\,|\partial\mathcal{D}|)^{-1} \cos\varphi \, dr \, d\varphi.$$

The dynamics of the billiard map are completely determined by the shape of the table. In rectangles and ellipses, for instance, the dynamics are completely integrable. Sinai introduced the

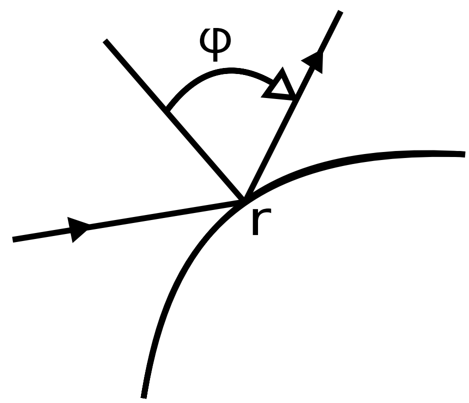

Figure 1: Billiard map coordinates

first class of chaotic billiards in 1970 [80]. In fact he showed that if $\partial\mathcal{D}$ is convex inwards and has no cusps then the system is hyperbolic, ergodic, mixing, and K-mixing. Since then many other classes of chaotic billiards have been studied, see for example the works of Bunimovich [20, 21, 22], Donnay [46], Markarian [69], and Wojtkowski [90].

A natural follow-up question for chaotic dynamical systems is: do they enjoy any nice statistical properties, in particular, do they behave at all similarly to independent identically distributed random variables? Showing that a system is ergodic is equivalent to showing that it obeys the strong law of large numbers, that is, for μ-integrable functions f we have

$$\int f d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^n x)$$

for μ-a.e. x. In other words, the time average of an integrable function is equal to its space average. This property has been proved for many types of billiards.

The central limit theorem is a stronger statement. It says that for large n the time average is distributed like a normal random variable with mean equal to the space average and with a variance which limits to zero as n grows to infinity. In some cases this holds for a smaller class of functions: those which are Hölder continuous on the collision space. The speed of the convergence of the variance depends on some factors we will explore in this chapter. We will examine some cases in which the aforementioned variance is of order $n^{-1}\ln n$, as opposed to the classical central limit theorem for i.i.d. random variables in which the variance is of order n^{-1}.

We will focus on billiards which are nonuniformly hyperbolic. Although the central limit theorem has previously been proved in some of the cases we investigate, we believe that our methods have two advantages: they can be applied to a wide variety of billiard systems and will give us a strong understanding of the variance of the normal distribution in the central limit theorem. To achieve this we will be utilizing the theorems developed in chapters four and five which rely largely on the application of the martingale central limit theorem to the problem at hand.

6.2 Some useful definitions and discussion

In order to discuss the central limit theorem in billiard dynamics we should recall some definitions. Let $f, g \in L^2(\mathcal{M}, \mu)$, then *correlations* are defined as

$$\mathcal{C}_n(f, g, T, \mu) = \int_{\mathcal{M}} (f \circ T^n) g \, d\mu - \int_{\mathcal{M}} f \, d\mu \int_{\mathcal{M}} g \, d\mu,$$

where T is the billiard map, μ is the invariant measure, and $\mathcal{M}$ is the previously defined collision space. The billiard map $T : \mathcal{M} \to \mathcal{M}$ is said to be *mixing* if for all $f, g \in L^2(\mathcal{M}, \mu)$ we have

$$\lim_{n \to \infty} \mathcal{C}_n(f, g, T, \mu) = 0.$$

The statistical properties of mixing systems can vary depending on the speed of convergence of $\mathcal{C}_n(f, g, T, \mu)$, which we often refer to as the rate of mixing. When determining the rate of mixing we consider functions f and g which are appropriately smooth; in general one considers functions which are Hölder continuous or piece-wise Hölder continuous, whose singularities coincide with those of the map T^k for some k. We say that correlations decay exponentially if

$$|\mathcal{C}_n(f, g, T, \mu)| < \text{const} \cdot e^{-an}$$

for some $a > 0$ and polynomially if

$$|\mathcal{C}_n(f, g, T, \mu)| < \text{const} \cdot n^{-b}$$

for some $b > 0$. The constant factors commonly depend on the functions f and g.

The billiards in this chapter are nonuniformly hyperbolic and have polynomial rates of mixing. This slow mixing rate causes the classical central limit theorem to fail. It is thus advantageous to construct a map induced by the billiard map which enjoys an exponential decay of correlations. In certain billiards this can be accomplished by considering a subset $M \subset \mathcal{M}$ in which we ignore "nonessential collisions;" these are sets in the phase space which contribute to the nonuniformity of the hyperbolicity. The classification of these collisions depends on the billiard table we are considering, so we will leave the specifics for subsequent sections. As in the previous chapter, we can define a return time function $\mathcal{R} : M \to \mathbb{N}$ by

$$\mathcal{R}(x) = \min\{m \geq 1 : T^m x \in M\},$$

and an induced billiard map $F : M \to M$ by

$$F(x) = T^{\mathcal{R}(x)}(x).$$

Furthermore, we define m-cells M_m as

$$M_m = \{x \in M : \mathcal{R}(x) = m\}.$$

This new dynamical system preserves the probability measure μ_M on M, where $\mu_M(A) = \mu(A)/\mu(M)$ for any $A \in M$.

Note that the collection of m-cells $\{M_m\}_{m=1}^{\infty}$ is an infinite partition of M into disjoint sets, each with positive probability. As discussed in Section 2.4 this allows us to directly compute certain conditional expectations. We will make extensive use of this in order to show that the central limit theorem holds for $\mathcal{R}$ in the billiard systems in the following sections.

The induced dynamical systems we study have exponential rates of mixing. Our goal in the following sections will be to exhibit non-classical central limit theorems for the observable $\mathcal{R}$ on various billiard tables, that is,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n \ln n}} \sum_{i=0}^{n-1} \left[\mathcal{R} \circ F^i - n\mathbb{E}(\mathcal{R})\right] \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where σ^2 is a constant which depends on the shape of the table being studied.

This is an amazing result on the tables we study for several reasons. It is possible for trajectories of the billiard to become stuck in arbitrarily long sequences of "nonessential collisions," that is, many iterations may occur in $\mathcal{M} \setminus M$. Consequently, the return time map $\mathcal{R}$ is unbounded in these systems. Furthermore, m-cells in these billiards have measure $\mu_M(M_m) \asymp m^{-3}$. This means that $\mathcal{R}$ also has infinite variance, as:

$$\mathbb{E}(\mathcal{R}^2) = \sum_{m=1}^{\infty} m^2 \mu_M(M_m) \asymp \sum_{m=1}^{\infty} m^{-1}.$$

An alternate way to interpret the central limit theorem is that, for large values of n, one can approximate the distribution of the average return time by

$$\frac{1}{n} \sum_{i=0}^{n-1} \mathcal{R} \circ F^i \sim \mathcal{N}\left(\mathbb{E}(\mathcal{R}), \frac{\sigma^2 \ln n}{n}\right).$$

Thus, even though $\mathcal{R}$ itself has infinite variance we find that the variance of the average of $\mathcal{R}$ over n iterations of the induced billiard map F is finite. For the tables we study it is already known by ergodicity that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{R} \circ F^i = \mathbb{E}(\mathcal{R}),$$

but the central limit theorem tells us that for large (but finite) numbers of iterations the average return time to the set M is typically finite and close to $\mathbb{E}(\mathcal{R})$.

Although a trajectory (under T) may become stuck in an arbitrarily large number of nonessential collisions, the central limit theorem indicates that this is highly atypical. However, this possibility does contribute to the nonstandard scaling factor found in the theorem. We see that the extra $\sqrt{\ln n}$ leads to the variance of the average being $\sigma^2 n^{-1} \ln n$ as opposed to the classical $\sigma^2 n^{-1}$. Clearly the variance in our case converges to zero more slowly, meaning that the convergence of the time average of $\mathcal{R}$ to its space average is also slower.

We note here that in the sections that follow we are presenting examples of applications that demonstrate that our main theorems hold, since in most cases the central limit theorem has been proved for the following billiards. This was done by Bálint and Gouëzel for stadia [9], by Bálint, Chernov, and Dolgopyat in the case of dispersing billiards with cusps [8], and by Szász and Varjú in the case of Lorentz gas with infinite horizon [83]. We believe that our method is applicable to a wide variety of systems and will be useful in determining relevant variances and diffusion coefficients in those systems; we intend to demonstrate this claim in subsequent sections.

6.3 A note on trapping trajectories

The billiards we study in the following sections are nonuniformly hyperbolic and have polynomial decay of correlations. This is due to portions of phase space which have very small or zero Lyapunov exponents and correspond to what we refer to as trapping trajectories. Speaking plainly, if a trajectory is close to a trapping trajectory then it stays close for a long time. This has a slowing effect on the dynamics and leads to the slow rate of mixing. As discussed in the previous section, it is advantageous to find a subset of phase space which does not see these trajectories and to define an induced return map on this subset.

We need to know what trapping trajectories look like in order to define the reduced space $M \subset \mathcal{M}$. One example occurs when a particle bounces between two parallel line segments for a large number of iterations, see Figure 2. An extreme case is a period-2 trajectory bouncing perpindicular to parallel sides of the table. A particle traveling in such a way clearly would never hit the circular obstacle in Figure 2. This type of billiard is often referred to as having infinite horizon.

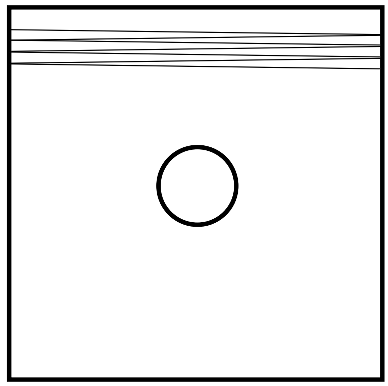

Figure 2: A trapping trajectory on a semi-dispersing billiard.

Another common trapping trajectory is the sliding trajectory. This occurs when a particle experiences a large number of near-tangential collisions on a circular arc, see Figure 3. Circular and elliptical billiards are known to be completely integrable, and thus a sliding trajectory is experiencing a period of non-chaotic motion.

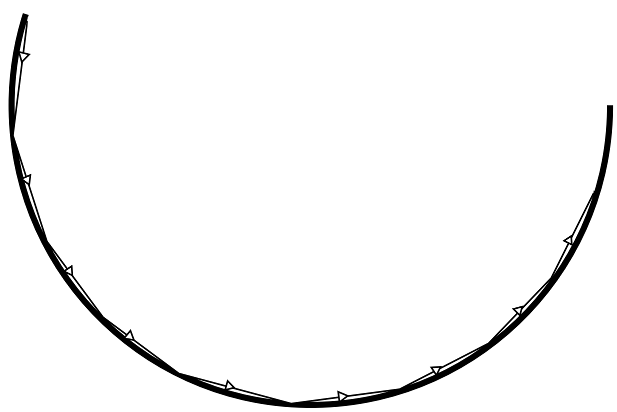

Figure 3: A sliding trajectory

Stadia and semi-dispersing billiards experience our first example of trapping trajectory, while stadia and skewed stadia both experience sliding trajectories.

6.4 Stadia

The stadium billiard table, introduced by Bunimovich in 1974 [19], is comprised of two equal semicircles which are connected by two parallel lines, see Figure 4. Dynamics on the stadium have been shown to be nonuniformly hyperbolic, ergodic, and mixing; for some discussion of these facts see [19, 21, 41]. Chernov and Zhang proved in [40] that correlations in stadia decay polynomially, in fact, they decay as $\mathcal{O}(1/n)$. As a consequence billiards in stadia do not satisfy the classical central limit theorem. However, in [9] Bálint and Gouëzel proved a non-classical version of the theorem which uses a scaling factor of $\sqrt{n \ln n}$ for Hölder continuous functions. We will restrict our attention to the return time map defined in Section 6.2.

Figure 4: A stadium billiard table

The induced billiard map on stadia is discussed extensively in [41]. We let $M \subset \mathcal{M}$ consist only of first collisions at focusing arcs and let the induced billiard map $F : M \to M$ and return time map $\mathcal{R} : M \to \mathbb{N}$ be defined as previously mentioned.

Theorem 6.1. *In stadia, the function $\mathcal{R}$ satisfies*

$$\frac{\sum_{i=0}^{n-1} \mathcal{R} \circ F^i - n\mathbb{E}(\mathcal{R})}{\sqrt{n \ln n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = const \cdot \frac{4+3\ln 3}{4-3\ln 3}$.

Proof. We begin by considering a truncated version of the return time function,

$$X_{n,i} = \left(\mathcal{R} \cdot \mathbf{1}_{M_{1,c_n}}\right) \circ F^i$$

where $c_n = \lfloor \sqrt{n} \ln \ln n \rfloor$ and $M_{1,c_n} = M_1 \cup \cdots \cup M_{c_n}$. It is in fact enough to show the central limit theorem for the truncated function. The m-cells M_m have the property that $\mu_M(M_m) \asymp m^{-3}$, and therefore satisfy (**H2**) from the previous chapter [41]. Due to this fact we have

$$\mu_M(\exists i \le n : F^i(x) \in M_{c_n,\infty}) = \mathcal{O}\left((\ln\ln n)^{-2}\right) \to 0$$

so that indeed, $(\mathcal{R}\cdot\mathbf{1}_{M_{c_n,\infty}})\circ F^i$ can be disregarded since its probability is negligibly small. What remains to be shown is that $X_{n,i}$ satisfies the assumptions of Theorem 5.7.

It is obvious from the definition that $|X_{n,i}| < c_n$. It follows from the measure of the m-cells that $\mathbb{E}(X_{n,i})$ is a constant and that

$$\mathbb{E}(X_{n,i}^2) = \sum_{m=0}^{c_n} m^2 \mu_M(M_m) = \mathcal{O}(\ln n)$$

. It is also known (see [25, 26, 39, 41], for instance) that the induced billiard system on stadia billiards satisfy assumptions (**h1**)-(**h3**) of the previous chapter, and therefore satisfy (**H1**) and enjoy exponential decay of correlations.

For the remainder of the theorem we must define an appropriate sequence of σ-algebras. We let $\mathcal{F}_{-1} = \{\emptyset, M\}$, $\mathcal{F}_0 = \sigma(\mathcal{R})$, and in general $\mathcal{F}_i = \sigma(\mathcal{R}, \dots, \mathcal{R}\circ F^i)$. By definition it is clear that the sequence $\{X_{n,i}\}$ is adapted to these σ-algebras and that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$.

It was shown in [25, 26] that if $x \in M_m$, then $Fx \in M_k$ for some $m/3 + o(1) \le k \le 3m + o(1)$ and that we have the transition probability

$$\mu_M(Fx \in M_k | x \in M_m) = \frac{3m}{8k^2} + \mathcal{O}\left(\frac{1}{m^2}\right)$$

for such cells. Therefore the stadium falls under the case of linear spreading described in Section 5.4 and all assumptions of Theorem 5.7 are satisfied. It follows that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n\ln n}} \sum_{m=0}^{n-1} [\mathcal{R}\circ F^m - n\mathbb{E}(\mathcal{R})] \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

where

$$\sigma^2 = c_0 \frac{4+3\ln 3}{4-3\ln 3} \tag{6.1}$$

and $c_0 = \lim_{n\to\infty} \dfrac{\mathbb{E}(X_{n,0}^2)}{\ln n}$, which follows from (5.35). □

6.5 Dispersing billiards with cusps

This class of billiards was first studied by Machta in [67]. It is known that the billiard map on these tables is hyperbolic and ergodic, however, the hyperbolicity is nonuniform. As a result,

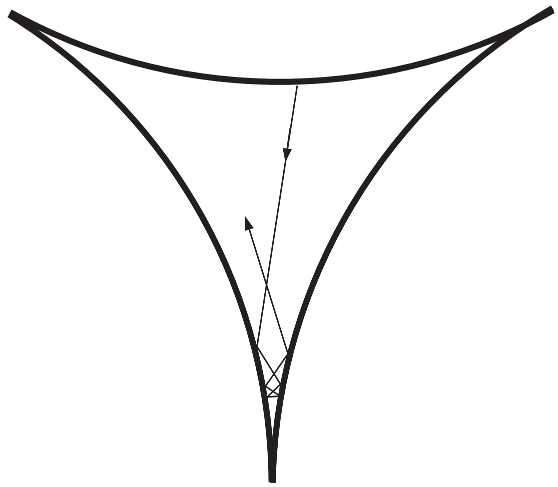

Figure 5: Machta's 3-cusp billiard table, courtesy [40]

correlations decay polynomially, in fact, Chernov and Zhang proved in [40] that the rate of mixing is $\leq \text{const} \cdot n^{-1}$. Recently it was shown by Bálint, Chernov, and Dolgopyat in [8] that a central limit theorem does hold in these systems for Hölder continuous functions; we again restrict our attention to the return time map.

On these tables the dynamics are slown when trajectories become trapped in a cusp for a large number of iterations. We wish to remove these collisions from our consideration, so if our table has i cusps at arclength paramter r_i, respectively, we construct the subset $M \subset \mathcal{M}$ by

$$M = \mathcal{M} \setminus \bigcup_i (r_i - \epsilon, r_i + \epsilon)$$

for some $\epsilon > 0$. Here m-cells M_m are made up of points in M whose subsequent collisions bounce in a cusp $m-1$ times. As was the case for stadia we have $\mu_M(M_m) \asymp m^{-3}$ and thus condition (**H2**) [37]. For this reason the return time map on the cusp billiard enjoys some of the same properties as that on the stadium billiard. Firstly, when proving the central limit theorem we can again consider the truncated function

$$X_{n,i} = \left(\mathcal{R} \cdot \mathbf{1}_{M_{1,c_n}}\right) \circ F^i$$

since the probability that trajectories enter higher-index cells is negligibly small. Furthermore, we again have that $\mathbb{E}(X_{n,i})$ is a constant and that $\mathbb{E}(X_{n,i}^2) = \mathcal{O}(\ln n)$.

The induced billiard map F on M was shown by Chernov and Markarian [37] to have exponential decay of correlations and satisfy (**H1**). Dynamics of F on billiards with cusps are markedly different than those on stadia though when it comes to travel between m-cells [40]. If $x \in M_m$ and $Fx \in M_k$ then

$$\mathcal{O}(\sqrt{m}) < k < \mathcal{O}(m^2),$$

and the transition probability from the m-cell to the k-cell is

$$\mu_M(Fx \in M_k | x \in M_m) \asymp \frac{m^{2/3}}{k^{7/3}}.$$

We need to be a little more precise. For concreteness, let us assume that $k \in (\sqrt{m}, m^2)$. Then we should have a constant C such that

$$C \sum_{k=\sqrt{m}}^{m^2} \frac{m^{2/3}}{k^{7/3}} = 1,$$

in this case that constant is $C = 4/3$. This gives a more accurate representation for the transition probabilities. Ignoring smaller order terms, we have

$$\mu_M(Fx \in M_k | x \in M_m) = \frac{4m^{2/3}}{3k^{7/3}}.$$

These cusp billiards exhibit algebraic spreading and thus we can apply Theorem 5.7. In order to be more specific with σ^2 we will go through the calculations on the relevant conditional expectations.

Let $\{\mathcal{F}_i\}_{i=-1}^{\infty}$ be σ-algebras defined as in the previous section. Again, by definition we know that $X_{n,i}$ is adapted to $\mathcal{F}_i$. Let us perform calculations as before. We have

$$\begin{aligned}
\mathbb{E}(X_{n,1}|\mathcal{F}_0) &= \sum_{m=0}^{c_n} \sum_{k=\sqrt{m}}^{m^2} k \mu_M(Fx \in M_k | x \in M_m) \mathbf{1}_{M_m} \\
&= \sum_{m=0}^{c_n} \sum_{k=\sqrt{m}}^{m^2} \frac{4m^{2/3}}{3k^{4/3}} \mathbf{1}_{M_m} \\
&= \mathcal{O}(\sqrt{X_{n,0}})
\end{aligned}$$

The last assumption of the theorem can also be shown by calculation:

$$\mathbb{E}(X_{n,1}^2|\mathcal{F}_0) = \sum_{m=0}^{c_n} \sum_{k=\sqrt{m}}^{m^2} k^2 \mu_M(Fx \in M_k | x \in M_m) \mathbf{1}_{M_m} \tag{6.2}$$

$$= \sum_{m=0}^{c_n} \sum_{k=\sqrt{m}}^{m^2} \frac{4m^{2/3}}{3k^{1/3}} \mathbf{1}_{M_m} \tag{6.3}$$

$$= 2X_{n,0}^2 + \mathcal{O}(X_{n,0}) \tag{6.4}$$

Theorem 6.2. *For the above billiard with cusps we have*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n \ln n}} \sum_{m=0}^{n-1} [\mathcal{R} \circ F^m - n\mathbb{E}(\mathcal{R})] \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = 2c_0 \tag{6.5}$$

and $c_0 = \lim_{n\to\infty} \frac{\mathbb{E}(X_{n,0}^2)}{\ln n}$.

Proof. As mentioned above, these billiard satisfy the necessary assumptions (**H1**)-(**H3**), so $\mathcal{R}$ must satisfy the CLT. Furthermore, (6.4) and Theorem 4.3 give the form for σ^2 in (6.5). □

6.6 Semi-dispersing billiards

Billiards in a square with a small fixed circular obstacle removed are known as semi-dispersing billiards. Chernov and Zhang proved [40] that this system has a decay of correlations bounded by const $\cdot\, n^{-1}$. Here the reduced phase space M is made up only of collisions with the circular object. The induced map $F : M \to M$ is then equivalent to the well studied Lorentz gas billiard map without horizon [39], which is known to have exponential decay of correlations and satisfy (**H1**), see [41]. It was proved by Szász and Varjú in [83] that a non-classical central limit theorem is satisfied in this billiard. The structure of the m-cells $M_m = \{x \in M : \mathcal{R}(x) = m\}$ is examined thoroughly in [25, 26, 41], we will use some of the facts presented in those references.

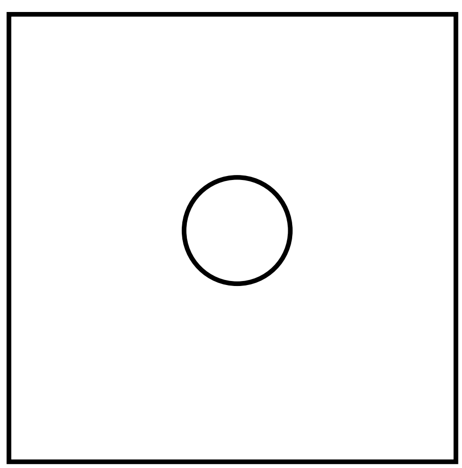

Figure 6: Semi-dispersing billiard table

Many properties of the m-cells and of the induced billiard map in the semi-dispersing case are quite similar to those in billiards with cusps. In particular, the measure of each m-cell is again $\mu_M(M_m) \asymp m^{-3}$ (so (**H2**) is satisfied), and as a result we once more have that the expectation of the return time map $\mathcal{R}$ is finite. It is also known that for a point $x \in M_m$ we have $Fx \in M_k$ where

$$\mathcal{O}(\sqrt{m}) < k < \mathcal{O}(m^2),$$

as in billiards with cusps. One major difference is the transition probabilities between cells; for dispersing billiards we have for admissable k that

$$\mu_M(Fx \in M_k | x \in M_m) \asymp \frac{m+k}{k^3}.$$

From this it is clear that semi-dispersing billiards are square spreading.

The assumptions of Theorem 5.7 are satisfied and it follows that the central limit theorem holds for $\mathcal{R}$. We will once more calculate the relevant conditional expectations.. For simplicity we make the same assumption as for billiards with cusps, that is, if $x \in M_m$ then $Fx \in M_k$ where

$$\sqrt{m} \leq k \leq m^2.$$

There is then a constant C that normalizes the transition probabilities, so that

$$C \sum_{k=\sqrt{m}}^{m^2} \frac{m+k}{k^3} = 1.$$

A simple calculation shows that $C = 2$. 3 some smaller order terms we have

$$\mu_M((Fx \in M_k | x \in M_m) = 2 \cdot \frac{m+k}{k^3}.$$

We have

$$\begin{aligned}
\mathbb{E}(X_{n,1} | \mathcal{F}_0) &= \sum_{m=0}^{c_n} \sum_{k=\sqrt{m}}^{m^2} k\mu_M(Fx \in M_k | x \in M_m)\mathbf{1}_{M_m} \\
&= \sum_{m=0}^{c_n} \sum_{k=\sqrt{m}}^{m^2} 2 \cdot \frac{m+k}{k^2}\mathbf{1}_{M_m} \\
&= \mathcal{O}(\sqrt{X_{n,0}})
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(X_{n,1}^2 | \mathcal{F}_0) &= \sum_{m=0}^{c_n} \sum_{k=\sqrt{m}}^{m^2} k^2\mu_M(Fx \in M_k | x \in M_m)\mathbf{1}_{M_m} & (6.6) \\
&= \sum_{m=0}^{c_n} \sum_{k=\sqrt{m}}^{m^2} 2 \cdot \frac{m+k}{k}\mathbf{1}_{M_m} & (6.7) \\
&= 2X_{n,0}^2 + \mathcal{O}(X_{n,0} \ln X_{n,0}) & (6.8)
\end{aligned}$$

Theorem 6.3. *The return time map on the above semi-dispersing billiard satisfies*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n\ln n}} \sum_{i=0}^{n-1} \left[\mathcal{R}\circ F^i - n\mathbb{E}(\mathcal{R})\right] \xrightarrow{d} \mathcal{N}(0,\sigma^2),$$

where

$$\sigma^2 = 2c_0 \tag{6.9}$$

and $c_0 = \lim_{n\to\infty} \frac{\mathbb{E}(X_{n,0}^2)}{\ln n}$.

Proof. It was shown above that this billiard is algebraic spreading and satisfies Theorem 5.7. In particular, (6.8) and Theorem 4.3 guarantee the form for σ^2 in (6.9). □

6.7 Skewed stadia

We now turn our attention to the skewed (or drivebelt) stadia. These tables, unlike the previously studied "straight" stadia, are constructed by connecting a major arc and a minor arc by two straight lines, rather than by connecting two semicircles, see Figure 7. These were introduced by Bunimovich in [19], where he also established their hyperbolicity and ergodicity.

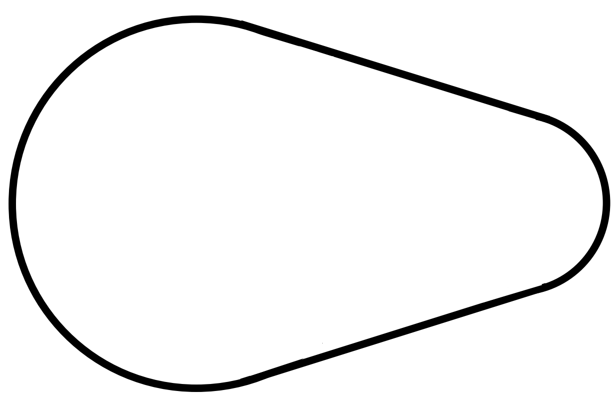

Figure 7: Skewed stadium

More recently, Chernov and Zhang [40] proved that skewed stadia have polynomial decay of correlations. In [39] the aforementioned authors also described in detail the space M and the structure of the associated m-cells. We will use facts from both works in our analysis.

As before, we can first consider the phase space $\mathcal{M}$ to be made up only of collisions with the arcs by unfolding the table, see Figure 8. Unlike straight stadia, using this method produces

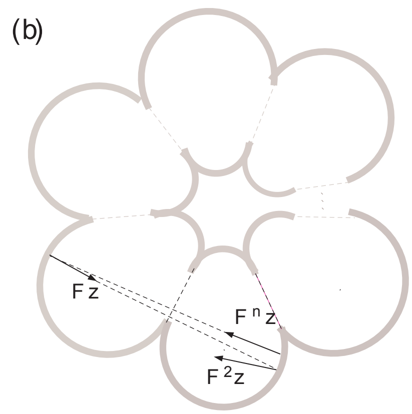

Figure 8: Unfolded skewed stadium, courtesy [39]

billiards with finite horizon since the lines used to connect arcs are not parallel. We let M be the set of all first collisions with a given arc, so that M_m is made up of points which collide with the same arc m times.

The skewed stadium enjoys many of the same properties as the straight stadium, for instance, the induced billiard map F has exponential decay of correlations (**H1**) and the measure of an m cell is $\mu_M(M_m) \asymp m^{-3}$ (**H2**). However, for a point $x \in M_m$ we have that $Fx \in M_k$, where

$$\frac{1}{7}m - \mathcal{O}(1) \leq k \leq 7m + \mathcal{O}(1).$$

Although points still spread linearly in m, we see that they can travel further than those in straight stadia. This wider range affects the normalizing constant present in the transition probabilities between cells, and we have

$$\mu_M(Fx \in M_k | x \in M_m) = \frac{7m}{48k^2} + \mathcal{O}\left(\frac{1}{m^2}\right). \tag{6.10}$$

Therefore skewed stadia are linear spreading, and we have the following thoerem.

Theorem 6.4. *For the drivebelt stadium, $\mathcal{R}$ satisfies*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n \ln n}} \sum_{i=0}^{n-1} \left[\mathcal{R} \circ F^i - n\mathbb{E}(\mathcal{R})\right] \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

where

$$\sigma^2 = c_0 \frac{24 + 7\ln 7}{24 - 7\ln 7} \tag{6.11}$$

and $c_0 = \lim_{n\to\infty} \frac{\mathbb{E}(X_{n,0}^2)}{\ln n}$.

Proof. In this section we have shown that the skewed stadium satisfies the assumptions of Theorem 5.7. This, along with (6.10), indicates that σ^2 has the form given in (6.11). $\square$

6.8 Linked twist map

We consider the two-dimensional torus $\mathbb{T}^2 = [0,2] \times [0,2]$ with coordinates (x, y) (mod 2). On this torus we define two overlapping annuli P, Q by $P = [0,2] \times [0,1]$, $Q = [0,1] \times [0,2]$. We denote the union of the annuli by $R = P \cup Q$ and the intersection by $M = P \cap Q$. The annuli P and Q are vertical and horizontal strips in the torus. In order to define a linked twist map on the torus we first define a twist map on each annulus. A twist map is simply a map in which the orbits move along parallel lines, but with a uniform shear. In particular, we define $F : R \to R$, such that

$$F(x,y) = \begin{cases} (x+2y, y), & \text{if } (x,y) \in P; \\ (x,y), & \text{if } (x,y) \in R \setminus P. \end{cases}$$

Note that F leaves points in $R \setminus P$ unchanged, and any horizontal line in P is invariant. We define the map G similarly:

$$G(x,y) = \begin{cases} (x, y+2x), & \text{if } (x,y) \in Q; \\ (x,y), & \text{if } (x,y) \in R \setminus Q. \end{cases}$$

Now the linked twist map H is defined by $H := G \circ F$, which maps from R to R. By calculating DH, one can easily check that $\det DH = 1$, which implies that H preserves the Lebesgure measure m on M.

As above, one can define a reduced map which enjoys exponential decay of correlations. More precisely, we define $F_M : M \to M$ to be the return map with respect to F, such that for any $(x, y) \in M$, $F_M(x, y) = F^n(x, y)$, where $n = \mathcal{R}_F(x, y)$ is the first return time of (x, y) to M under iterations of F. Similarly, we define $G_M : M \to M$, such that $G_M(x, y) = G^n(x, y)$, where $n = \mathcal{R}_G(x, y)$ is the first return of (x, y) to M under iterations of G. We define the reduced map as $T := G_M \circ H_M$. Then H_M is the first return map obtained from H onto M. Note that G is an Anosov diffeomorphism restricted on M, so by the uniformly hyperbolicity of G on M there exists $N = N(G) > 1$ such that $G^N M \subset M$. Let m_M be the conditional Lebesgure measure on M, then T preserves m_M.

Let $S_{\pm 1}$ be the singular set of the reduced map $H^{\pm 1}: = H_S^{\pm 1}$. In [82], Figure 2 shows the structure of S_1 while Figure 5 shows the image of S_{-1}. Using the notation of that paper, we label by $\{\Sigma_n\}$ the connected regions near $(1,0)$ in S_1, as shown in Figure 6, on which the return time function is n. We know from Appendix A of [82] that the cell Σ_n has length of order $1/n$ and width of order $1/n^2$. Similarly, we denote by $\{M_n\}$ cells in S_{-1} with backward return time n. As it was shown in Lemma 5.3 of [82], unstable manifolds have slope $1+\sqrt{2}$, thus we know that the longer boundary curves of M_n all have slope approximately $1+\sqrt{2}$, and these cells converging to the fixed point $(1,0)$ as $n \to \infty$. In addition, one can show that M_n has length of order $O(n^{-1})$ and width of order $O(n^{-2})$. In the proof of Lemma 5.4 of [82], it was shown that when an unstable manifold W intersects Σ_n for some n large enough, it only crosses those Σ_m with $m \in [n, (3+2\sqrt{2})n]$. If we redefine n then we can say that W intersects only cells Σ_m, with $m \in I_n = [n/\beta + c_1, \beta n + c_1]$, where $\beta = 1+\sqrt{2}$, for some constants c_1, c_2.

In terms of the singular set S_{-1}, this implies that the image of $\partial M_n \subset S_{-1}$ will only intersect Σ_m, for $m \in I_n$, i.e. $M_n \subset \cup_{m \in I_n} \Sigma_m$. Thus we take an unstable manifold W that completely stretches across M_n, then its image HW will be cut into pieces such that each piece is stretched completely across M_m, for $m \in I_n$.

Note that for large n, the region $M_n \cap \Sigma_n$ is nearly a rectangle with dimension $O(m^{-2}) \times O(n^{-2})$. Now Lemma 5.2 in [82] implies that the expansion factor of unstable manifolds in Σ_m is $O(m)$, thus $TM_n \cap \Sigma_m$ is a strip in M_m that is completely stretched in the unstable direction and has width $O(\frac{1}{mn^2})$.

Thus one can now check that the transition probability of moving from Σ_n to Σ_m is

$$\frac{\mu(\Sigma_m \cap TM_n)}{\mu(TM_n)} = \frac{c\frac{1}{m^2n^2}}{\frac{1}{n^3}} = c_0 \frac{n}{m^2}$$

where $c = \left[\beta - \beta^{-1}\right]^{-1}$ is the normalizing constant, such that $\sum_{m \in I_n} \mu(\Sigma_m \cap TM_n) = \mu(TM_n)$. From this it follows that the linked-twist system has the linear spreading property detailed in Section 5.4, and by Theorem 5.7 it follows that we have the following:

Theorem 6.5. *The return time function in the linked twist system satisfies*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n \ln n}} \sum_{i=0}^{n-1} \left[\mathcal{R} \circ H^i - n\mathbb{E}(\mathcal{R})\right] \xrightarrow{d} \mathcal{N}(0, \sigma^2) \tag{6.12}$$

where

$$\sigma^2 = c_0 \frac{\beta - \beta^{-1} + 2\ln\beta}{\beta - \beta^{-1} - 2\ln\beta} \tag{6.13}$$

with $\beta = 1 + \sqrt{2}$ *and* $c_0 = \lim_{n\to\infty} \frac{\mathbb{E}\left[(\mathcal{R}|_{\{\mathcal{R} < \sqrt{n}\ln\ln n\}})^2 \right]}{\ln n}$

Proof. As mentioned above, this system is linear spreading, and the transition probability along with (5.35) guarantee the form of σ^2 given in (6.13). $\square$

CHAPTER 7

Ergodicity and coexistence of elliptic islands in a family of convex billiards

This chapter is joint work with Jingyu Chen, Hong-Kun Zhang, and Pengfei Zhang, to appear in *Chaos*. We study a two-parameter family of convex billiard tables constructed by taking the intersection of two round disks in the plane. We conjecture that certain families of these tables are chaotic and support this claim with numerical and theoretical results.

7.1 Introduction

Billiard systems are a class of dynamical systems originating in statistical mechanics, in which a particle moves freely along strict segments in a bounded region in the plane (which is called the billiard table), and changes its velocity according to the law of elastic reflection upon collisions with the boundary of the billiard table. The dynamics of the billiard systems are determined by the shapes of the tables, and may vary greatly from regular (completely integrable) to strongly chaotic behaviors.

The study of billiard systems was pioneered by Sinaĭ[80] in his seminal paper on *dispersing billiards*, where he proved the hyperbolicity and ergodicity of these system and derived various statistical properties. The mechanism for the hyperbolicity of dispersing billiards is that, dispersing wave-fronts remain dispersing after each collision. In 1974 Bunimovich [19] discovered the *defocusing mechanism* of convex billiard tables, and proved the hyperbolicity and ergodicity of stadium billiards. A convex table may also be hyperbolic if the focusing wavefronts spend enough time on defocusing. See also [22, 46, 69, 90] for various improvements of the defocusing mechanism and new ergodic tables.

There are only a few model of billiards that fail this mechanism and are known to be fully chaotic. In [11] Benettin and Strelcyn introduced the *oval tables* and observed the bifurcation phenomena, the coexistence of elliptic and chaotic regions, and the separation of the chaotic region into several invariant components. Moreover they gave numerical estimates of the Lyapunov exponent and the entropy of these billiard dynamical systems. In [10] the authors studied a two-parameter family of convex billiard tables, the *squash billiard tables*, on which the defocusing mechanism does not take place. They gave a numerical and a heuristic proof of the ergodicity of squash billiards. For more related discussions see [54, 56, 65, 66].

Another family of convex tables, the *lemon shaped tables*, was introduced by Heller and Tomsovic [55] in 1993, by taking the intersection of two unit disks. The coexistence of the elliptic islands and chaotic region has also been observed numerically in [68, 73] for most lemon tables. The only possible exception is when the centers lie on each others' boundaries, which is the starting point of our study. In fact, we put the lemon tables under a more general family, among which the ergodicity may no longer be an exceptional phenomenon. Our tables are also simple, obtained by intersection of a unit disk D_1 with another round disk D_R with radius $R \geq 1$, see Fig. 9, where B measures the distance between the centers of these two disks. Since the boundary of our billiard table consists of two circular arcs this makes the billiard systems much easier to study. At the same time, these systems already exhibit rich dynamical behaviors. We have found that there exists an infinite strip $\mathcal{D} \subset [1, \infty) \times [0, \infty)$, such that for any $(R, B) \in \mathcal{D}$, the billiards on $Q(R, B)$ is ergodic.

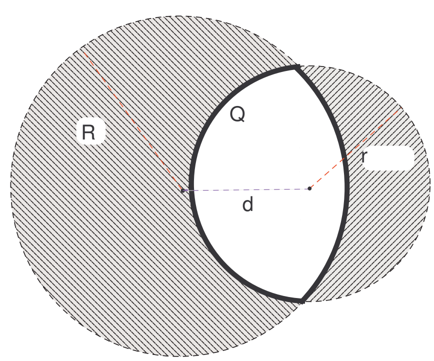

Figure 9: Basic construction of our billiard table $Q(R, B)$

Taking the degenerate case when $R = 1$ and $B = 0$, one can check that $Q(1, 0)$ is a unit disk table, which is completely integrable. On the other hand, letting $R \to \infty$, the limit cases are various tables obtained by cutting a disk by a straight line (see also [73]). It is well known that the resulting table is hyperbolic and ergodic if and only if the curved boundary component is a major-arc. Note that the billiard table is degenerate or empty if $B \geq R+1$, as the two disks do not intersect each other. Similarly, $Q(R,B) = D_1$ if $B \leq R-1$, since $D_1 \subset D_R$. In the second case, there are also interesting dynamics if we set the billiard table to be the annulus between the two disks: $A(R,B) = D_R \backslash D_1$ (when $R > B+1$). In fact, this *annulus table* has already been studied extensively [5, 50, 75] from the 80's. So in this paper, we will focus on the family of tables $Q(R,B)$ with the parameter space

$$\Omega = \{(R,B) \subset (1,\infty) \times (0,\infty) \, : \, -1 < B - R < +1\},$$

which contains new billiard tables that have never been studied before. Fig. 10 clarifies the regions in the first quarter $(R,B) \in [0,\infty) \times [0,\infty)$, where $\Omega' = \{(R,B) \subset [0,1] \times (0,\infty) \, : \, -1 < B - R < 1\}$ refers to an equivalent class of billiards as those in Ω (by switching the roles of r and R). There are three regions in $[0,\infty) \times [0,\infty)$: $B > R+1$ refers to the degenerate case; for $B < R-1$, also degenerate, and interesting dynamics happens in the annulus table; Ω' and Ω refer to two equivalent families of 2-parameter convex billiards. The subregions I,II and III in Ω are characterized by the relative positions of the two centers with respect to the table $Q(R,B)$.

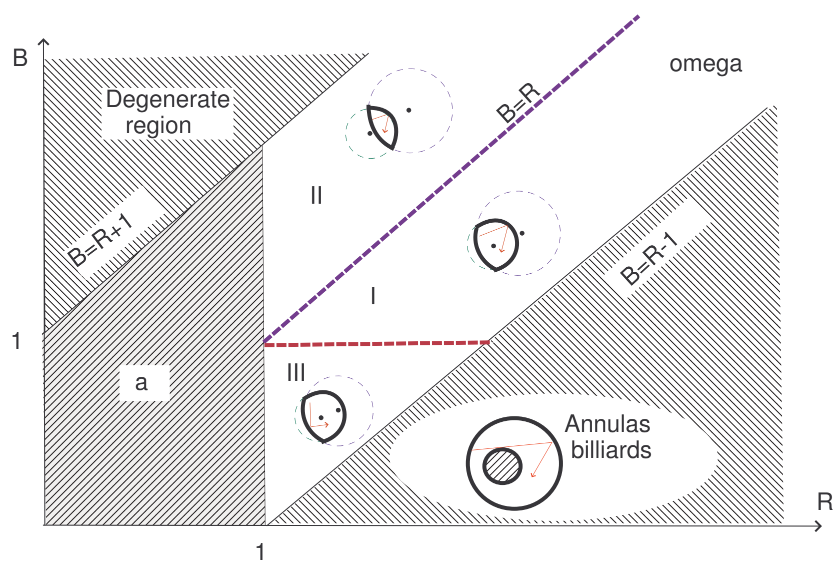

Figure 10: Regions in $[0,\infty) \times [0,\infty)$

It is observed in [75] that the phase space of the annulus billiard $A(R,B) = D_R \backslash D_1$ is divided into three subregions (when $0 < B < R-1$): the *completely integrable region* in which the billiard trajectories never hit the inner circle; the *nearly integrable region* in which the billiard trajectories have strictly alternative collisions between the inner and outer circles; the *chaotic region* in which the collisions of the trajectories 'randomly' alternate with the inner and outer circles. Recently, Bunimovich [27] constructed the first class of natural and visible systems with coexistence phenomena: mushroom billiards, which combine the completely integrable dynamics on elliptic table and the completely chaotic dynamics on elliptic stadium. In [68, 73], the authors also observed numerically the bifurcations and coexistence of chaotic regions and elliptic islands on most lemon billiards, when the distance B is either too large ($B > 1$) or too small ($B < 1$). Similar phenomena also appear in our tables $Q(R, B)$ for $R > 1$, when B is either too large ($B > R$) or too small ($B < 1$). Our billiard systems also supply simple examples that exhibit the coexistence phenomenon.

Now we conclude the following observation from our various numerical simulations:

Observation. *For parameters in a larger set of Region I, the billiard tables $Q(R, B)$ are ergodic.*

This is demonstrated in Fig. 11. The small solid region contains parameters with which the billiards have elliptic islands, and the parameters undergoes deformations and disappear when the parameters leave the solid region. The rest of parameters in the shaded Region I correspond to tables with the ergodic property (according to our simulation). The dotted curve in Region I has equation $B = \sqrt{R^2 - 1}$. The segment over $R = 1$ corresponds to the one-parameter family lemon tables.

Intuitively, our observation says that the billiard system is likely to be ergodic if the distance between the two centers takes the intermediate values $1 \leq B \leq R$. Note that the limit parameter $B \leq R \to \infty$ corresponds to the major-arc table, which is known to be hyperbolic and ergodic. Moreover, we also propose the following conjecture:

Observation. *There exists $R_0 \gg 1$, such that for all $R > R_0$, billiards on the table $Q(R, B)$ is ergodic provided that $B \in (1, \sqrt{R^2 - 1})$.*

The geometric meaning of the condition $B \in (1, \sqrt{R^2 - 1})$ is that the boundary of the table contains a major arc. In fact, the tables in Section 7.3.3 can also be viewed as *small* perturbations of the major-arc table: the table obtained by simply closing a major arc by a straight line segment

AD. Denote such a table by Q_0. It is well known that the billiard dynamics on Q_0 is equivalent to that of a table with a boundary consisting of two identical major arcs and hence satisfies Bunimovich's defocusing mechanism. So the dynamics on the table Q_0 is hyperbolic and ergodic. We then alter the curvature of the curve connecting A and D by varying the radius R (then the center distance B changes accordingly). Although an arbitrary small perturbation can make the table fail the defocusing mechanism, but our simulation shows that the ergodicity may survive under these small perturbations, as long as the table continues to satisfy the condition $1 \ll B < R$.

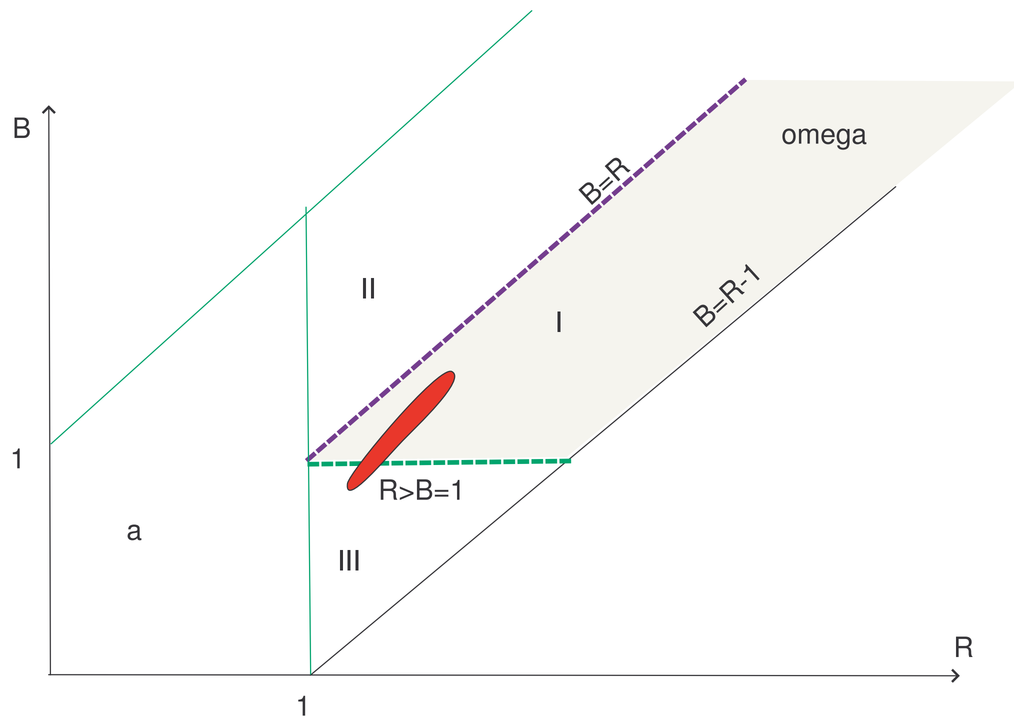

Figure 11: Red region: there exists an elliptic period-3 orbit for each table in this region, and we observe a small elliptic island surrounding it, which clearly destroys the ergodicity.

The paper is organized as follows: in Section 2 we give a brief introduction of general billiard systems and some features of our billiard table $Q(R, B)$. In Section 7.3 we first study a special table $Q(1, 1)$ and verify its ergodicity numerically. Then we examine three different types of perturbations of the billiard table $Q(1, 1)$ with the three parameters satisfying $1 < B < R$ (Region I), $1 < B = R$ (the boundary of Region I along the diagonal), and $1 = B < R$ (the boundary at the bottom of Region I). We observe the ergodicity of billiard tables with parameters in a large set in these family, and also detect tables with a small region of parameters among which the ergodicity fails due to the existence of elliptic periodic orbits. In each subsection we examine the dominating periodic orbits and their effects on the dynamics. In Section 5 we study the non-ergodic perturbations in Region Ⅱ and Ⅲ and observe the bifurcation of periodic orbits and the generation of elliptic islands surrounding them. In Fig. 11 we summarize the conclusions obtained in this study.

Although parts of our results are only based on numerical simulations, a rigorously mathematical justification is currently under investigation. The most difficult step is to prove hyperbolicity, especially since the classical defocusing mechanics fails in our model, and it is not obvious that the hyperbolicity can be guaranteed by considering any fixed higher iterations. Instead, our preliminary calculation shows that one should define a stopping time function τ and the associated induced map $F(x) = T^{\tau(x)}x$. By properly choosing the stopping time it is possible that the induced map enjoys hyperbolicity as well as ergodicity.

7.2 Preliminaries

In this section we first introduce the notations of billiard systems and then describe some basic properties of our billiard table. Let Q be a compact convex domain in the plane, $\Gamma = \partial Q$ be the boundary of Q equipped with the arc-length parametrization. The phase space of the billiard system on Q is a cylinder $M = \Gamma \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. A point $x \in M$ has the coordinate representation $x = (s, \varphi)$, where s is measured by its arc-length along the oriented boundary Γ, and φ is the angle measured from the inner normal direction to the outgoing velocity vector after the reflection. The billiard map $T : M \to M$ sends a point (s, φ) to the point (s_1, φ_1) right after its next collision with Γ. The derivative DT at the point $x = (s, \varphi)$ is denotes as $D_{(s,\varphi)}T$, which is given by (see [41, (2.26)]):

$$\frac{-1}{\cos\varphi_1} \begin{pmatrix} \tau K + \cos\varphi & \tau \\ \tau K K_1 + K\cos\varphi_1 + K_1\cos\varphi & \tau K_1 + \cos\varphi_1 \end{pmatrix} \tag{7.1}$$

where $(s_1, \varphi_1) = T(s, \varphi)$, K is the curvature of radius of Γ at $\Gamma(s)$, and K_1 is the curvature of radius of Γ at $\Gamma(s_1)$. Moreover, T preserves a natural measure $d\mu = c \cdot \cos\varphi \, ds \, d\varphi$, where c is a normalizing constant. See [41] for more information.

Now we consider a 2-parameter billiard table $Q(R, B)$, obtained by intersecting a unit disk D_1 with a larger disk D_R of radii R, with the distance B between their centers. As noted in the

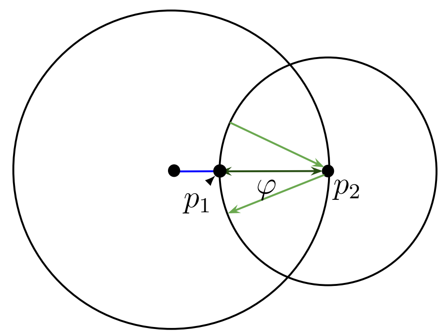

Figure 12: A family of period-4 orbits with $B = 1$ or R.

introduction, we always assume that $R > 1$ and $R - 1 < B < R + 1$. There are two corners on the table, which break down the smoothness of the boundary Γ and will lead to the existence of nontrivial singularity curves. More precisely, the singularity set of this table $Q(R, B)$ consists of two vertical segment in M based at these two corner points, as well as the horizontal lines $\varphi = \pm\pi/2$.

We first note that for the billiard dynamics on each table $Q(R, B)$, there exists exactly one periodic orbit $\mathcal{O}_2$ of period 2, which bounces perpendicularly between the midpoints $\{p_1, p_2\}$ of the circle arcs (see Fig. 12). The coordinate representation of this period-2 orbit is given by $\mathcal{O}_2 = \{(p_1, 0), (p_2, 0)\}$. Recall that [68] a periodic point $x = T^k x$ is said to be hyperbolic, parabolic and elliptic if $|\mathrm{Tr}(D_x T^k)| > 2$, $|\mathrm{Tr}(D_x T^k)| = 2$ and $|\mathrm{Tr}(D_x T^k)| < 2$, respectively.

Proposition 7.1. *Let $\mathcal{O}_2$ be the period-2 orbit of the billiard map on the table $Q(R, B)$. Then this orbit is hyperbolic if $1 < B < R$, is parabolic if $B = 1$ or $R = B$, and elliptic if $B < 1$ or $B > R$.*

Proof. This proposition is proved [68, 73] in the case $R = 1$, by calculating the trace of tangent map. Here we use the same approach. Note that the travel time τ between the collisions satisfies $\tau + B = 1 + R$, $K_{p_1} = -1$ and $K_{p_2} = -1/R$. Then after a simple calculation, the trace of the derivative $D_{(p_1,0)}T^2$ is (by Eq. (7.1))

$$\begin{aligned}\mathrm{Tr}(D_{(p_1,0)}T^2) =& 4(1-\tau)(1-\tau/R) - 2\\ =& 4\left(1 - \frac{B}{R}\right)(1 - B) - 2.\end{aligned}$$

So there are three different qualitative behaviors of the periodic orbits:

1. If $1 < B < R$, then $\mathrm{Tr}(D_{(p_1,0)}T^2) < -2$, hence $\{(p_1, 0), (p_2, 0)\}$ is a hyperbolic periodic orbit.

2. If $B = 1$ or $B = R$, then $\mathrm{Tr}(D_{(p_1,0)}T^2) = -2$, hence $\{(p_1, 0), (p_2, 0)\}$ is a parabolic orbit.

3. If $R - 1 < B < 1$ or $R < B < R + 1$, then $-2 < \mathrm{Tr}(D_{(p_1,0)}T^2) < 2$, hence $\{(p_1, 0), (p_2, 0)\}$ is an ellipitic orbit.

This finishes the proof of the proposition. □

7.3 Ergodic tables in Region I

In this section we will investigate the ergodicity of tables with parameters in Region I. From Fig. 10 we know that Region I is an unbounded strip with three line boundary components. Clearly, for parameters on the line segment $B = R-1$, $Q(R, B)$ represents the unit disk D_1 and the dynamics is completely integrable. To investigate the boundary $1 = B < R$ as well as $1 < B = R$, we first start with a special case of our two-parameter family, the table with $B = R = 1$. This table $Q(1, 1)$ has been well studied (see [68, Fig. 4(f)] after setting their parameter $w = 0.5$, and also in [73, Fig. 4(e)] after setting their parameter $\delta = \delta_c$). In particular, it is observed that this table is indeed ergodic by numerical simulations (see also Fig. 13, in which we demonstrate the iterations along one typical phase point $(1.5, 0.01)$ after $100,000$, $1,000,000$ and $100,000,000$ iterations, respectively. Our numerical results show that, visually, the phase space is completely filled after $100,000,000$ iterations. Recall that a simple criterion for a measure-preserving system (X, μ, T) to be ergodic is, the averages $\frac{1}{n}\sum_{k=0}^{n-1}\delta_{T^k x}$ has the same limit distribution for μ-a.e. $x \in X$. We also tried quite a few of different initial points and get the same asymptotic distribution for large enough iterations. So the billiard dynamics on $Q(1, 1)$ should be ergodic.

To get a better understanding of the dynamical systems, it is often rewarding to study its statistical properties. The starting point of this investigation is the *decay of correlations*. Recall that the billiard map T preserves a natural measure $d\mu = c\cos\varphi\, ds\, d\varphi$, where c is the normalizing constant. Then the correlation for two functions $f, g \in L^2(\mu)$ is defined as:

$$C_{f,g}(n) = \int f \cdot (g \circ T^n) d\mu - \int f d\mu \int g d\mu.$$

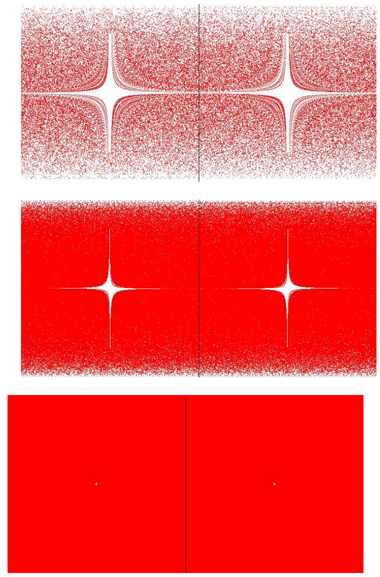

Figure 13: Trajectory segments with an initial point $(1.5, 0.01)$ on the phase space of the table $Q(1,1)$, after $100,000$, $1,000,000$ and $100,000,000$ iterations, respectively.

A measure-preserving system is said to have exponential decay of correlations if there is a constant $a > 0$ such that $|C_{f,g}(n)| \leq c_{f,g}e^{-an}$ for all $n \geq 1$ and for any Holder observables f, g on M, where $c_{f,g} > 0$ may depends on f and g. Now it is well known that uniform hyperbolic systems and dispersing billiard systems have exponential decay of correlations. The situation may be rather complicated for convex billiard systems, since the system may only have polynomial decay of correlations. That is, there is a constant $a > 0$ such that $|C_{f,g}(n)| \leq c_{f,g}n^{-a}$ for all $n \geq 1$ and for any Holder observables f, g on M. In fact, slow decay of correlations [39, 40, 70] has already been carried out for several classes of chaotic billiards including semidispersing billiards, Bunimovich-type billiards and Bunimovich stadia. We believe for the decay rates of billiard systems constructed in this paper, their general scheme should still work.

For our purposes we take the position function as the observable, that is, the projection of a point $x = (s, \varphi)$ to its first coordinate s. The correlations $C_{s,s}(n)$ give us an idea of the relationship between the initial position s_0 and the position s_n after n-th iterations under the billiard map T. To support our observation that $Q(1,1)$ is an ergodic (even mixing) table, we further computed the decay rate of the position function: $\lim_{n\to\infty} \log(C_{s,s}(n))/\log(n)$ (see Fig. 14). We can see that the limit $\log(C_{s,s}(n))/\log(n)$ converges to -0.28 as $n \to \infty$. Therefore, the correlation function $C_{s,s}(n)$ decays at the rate of $\frac{1}{n^{-0.28}}$. In fact, this power-law decay of correlation is quite common if the system admits many parabolic periodic orbits and hence suffers the stickiness effect caused by these orbits. We will have a detailed discussion below.

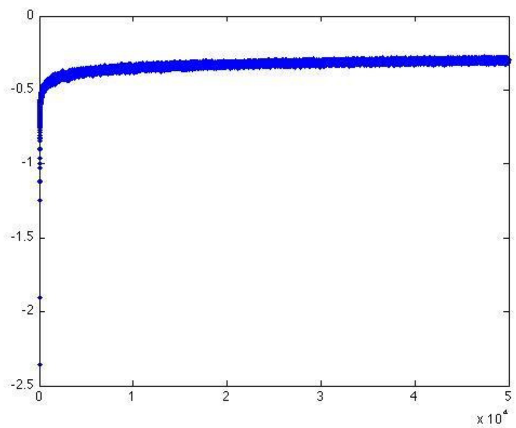

Figure 14: Graphs of the correlation function $n \mapsto \log(C_{s,s}(n))/\log(n)$ for table $Q(1,1)$

Beside the existence of the period-2 orbit described in Proposition 7.1, there are four segments in the phase space of this table which are fixed by the fourth iterate T^4 (see Fig. 12). They are: $\{(s,0) : s \text{ lies on } C_1\}$, $\{(s,0) : s \text{ lies on } C_2\}$, $\{(p_1,\varphi) : \varphi \in (-\pi/3,\pi/3)\}$ and $\{(p_2,\varphi) : \varphi \in (-\pi/3,\pi/3)\}$. Then a simple calculation shows the following (see also [68, 73]).

Proposition 7.2. *Let $Q(B,B)$ be the table with $B>1$. Every periodic orbit $\mathcal{O}_4$ in above families is parabolic.*

Proof. Let us start with a periodic orbit $\mathcal{O}_4$ given by $(p_2,\varphi) \to (s_\varphi,0) \to (p_2,-\varphi) \to (-s_\varphi,0) \to (p_2,\varphi)$. The rest orbits in these families can be treated similarly. Note that the travel time τ between each collision of this orbit is exactly 1. By (7.1), the tangent map of T^4 at this orbit is given by

$$D_{(p_2,\varphi)}T^4 = \left(\frac{1}{\cos\varphi}\begin{pmatrix} -1+\cos\varphi & 1 \\ -\cos\varphi & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\cos\varphi & -1+\cos\varphi \end{pmatrix}\right)^2$$

$$= \begin{pmatrix} -1 & 2-2\cos\varphi \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 4\cos\varphi - 4 \\ 0 & 1 \end{pmatrix}.$$

So $\mathrm{Tr}(D_{(p_2,\varphi)}T^4) = 2$ and such a periodic orbit is parabolic. $\square$

One topic of great interest in the study of billiard systems is the existence of parabolic periodic orbits and their stickiness effects on the billiard dynamics. Recall that ergodicity requires that the asymptotic distribution of a typical orbit segment converges to the smooth invariant measure.

But the speed of the convergence could be slow down significantly, if the orbit runs close to some sticky periodic orbits, since the trajectory will become trapped by these parabolic periodic orbits for a long time. It can be easily seen from Fig. 13 that the trajectory approaches very slowly to thees periodic points, because once it comes close, it has to stay close to $\mathcal{O}_4$ for a long time. This kind of orbits also exist in the annulus billiards, which have a significant effects on the dynamics. More precisely, on any annulus billiard table $A(R, B) = D_R \backslash D_1$ with $0 < B < R - 1$, there exist infinite families of parabolic periodic orbits whose trajectories avoid the inner circle but intrude into the *influence disk* D_{B+1} of the inner disk. These are the so called Marginally Unstable Periodic Orbits (MUPO for short, see [2] for more details), which have a major effect on the ergodic properties and decay of correlations via *sticking* nearby orbits on the annulus table for a long time. Also see [5] for a detailed discussion of the stickiness effect of the sticky periodic orbits in stadium-type billiard systems. So the periodic orbits $\mathcal{O}_4$ on the table $Q(B, B)$ serve as MUPOs of our billiard dynamics and should be responsible for the slow decay of the correlations of the billiard dynamics.

Next we will examine three different types of perturbations $Q(R, B)$ of the above billiard table $Q(1, 1)$, whose parameters satisfy $1 < B < R$ (corresponding to Region I in Fig. 10), as well as the line segments $1 < B = R$ and $1 = B < R$. Based on our observations, the billiard dynamics are ergodic on 'most' tables in these three cases. These results are particularly interesting because they provide us with some new ergodic convex billiard tables which fail the defocusing mechanism.

7.3.1 Tables with parameters on the boundary $1 < B = R$

In this subsection we investigate tables with parameters on the line segment $1 < B = R$, which is one of the boundary of Region I in Fig. 10. We increase the parameter $R = B$ of the table $Q(B, B)$ from 1 to ∞. As B goes to ∞, the limiting table $Q(\infty, \infty)$ will have the shape of a semidisk, on which the billiard dynamic is equivalent to the round disk table and hence is completely integrable. Moreover, due to Proposition 7.1, the periodic orbit $\mathcal{O}_2$ is parabolic on each table in this family. However, according to our numerical results, for any finite values of $B > 1$, these billiard systems appear to be ergodic, as we can see in Fig. 15 (with $B = 1.0101$) and Fig. 16 (with $B = 100$), for an initial point $(1.5, 0.1)$, after $100,000$, $1,000,000$ and $100,000,000$ iterations, respectively.

A special aspect of this family of tables is that, for any value of B, the center of the unit disk

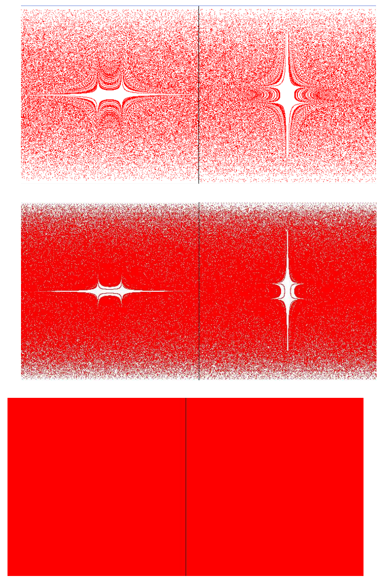

Figure 15: Trajectory segments on the phase space of the table with $R = B = 1.0101$.

is always located on the boundary of $Q(B, B)$. This leads to the existence of a family of periodic orbits of period 4. In fact, each trajectory emanating from the center of the smaller circle and hitting the boundary of D_1 will lead to one of the periodic orbits, say $\mathcal{O}_4$. Clearly the travel time τ between each collision equals to the radius $r = 1$. A similar calculation as in Propostion 7.1 shows that the trace is $\mathrm{Tr}(D_{(p_2,\varphi)}T^4) = 2$. Hence all periodic orbits $\mathcal{O}_4$ in this family are parabolic. As already mentioned, these orbits will cause a significant slowing down effect for the convergence of time averages and the decay rate of correlations.

Besides these parabolic periodic orbits, there exist another family of nonperiodic, but *sticky* orbits on these tables: the sliding trajectories, which will become dominant when the corners approximate an right angle. Recall that a trajectory is *sliding* if it collides almost tangentially at a circular arc for many consecutive occasions. As we mentioned in the beginning of this subsection, the table $Q(B, B)$ approaches a semi-disk when $B \to \infty$. So these sticky orbits will occur when the boundary of the table created by the larger circle becomes flat enough to sustain the sliding after the trajectory bounces off of that boundary. That is, after traveling a long time almost tangentially along the unit circle, the point finally reaches the larger circle, and bounces off of that boundary while keeping almost tangential to the unit circle, since the two pieces of the table are almost perpendicular to each other. So this trajectory will stay in one side of the phase space for a tremendous number of iterates and will only visit two small spots on the other side of the

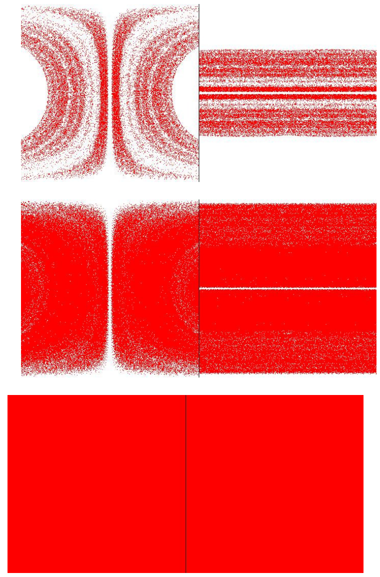

Figure 16: Trajectory segments on the phase space of the table with $R = B = 100$.

phase space once in a while. This sliding phenomenon also contributes to the significant slowing effect on the properties of billiard dynamics on $Q(B, B)$.

7.3.2 Tables with parameters on the boundary: $1 = B < R$

In this subsection we investigate the ergodic property of billiards with parameters lying on another boundary component $1 = B < R$ of Region I. We fix the distance $B = 1$ and let the radius R of the larger circle vary for all admissible values $1 < R < 2$. This corresponds to the common boundary of Region I and III in the parameter space, see Fig. 10. Numerically, we let $B = 1$ in the following simulations, and gradually increase R from 1.01 to 1.2. Then we pick an initial point $(1.5, 0.1)$, and run $50,000$, $300,000$ and $10,000,000$ iterations, respectively. As one can see in Fig. 17 and 18, the phase spaces of these tables are eventually filled by iterates along one trajectory, which gives us the hint that these tables should be ergodic. However, we also notice a short interval in $1.25 < R < 1.31$, in which the billiard systems admit an elliptic island, see Section 7.3.4 for further explanations.

A family of period-4 orbits similar to those in Fig. 12 survive on these billiard tables. It is easy to see that all these periodic orbits are also parabolic, and will have the stickiness effect on the dynamics. Moreover, a new family of parabolic periodic orbits become dominant when R gets close to $B + 1 = 2$. Recall that the periodic orbits are dense on the round disk table, and all of

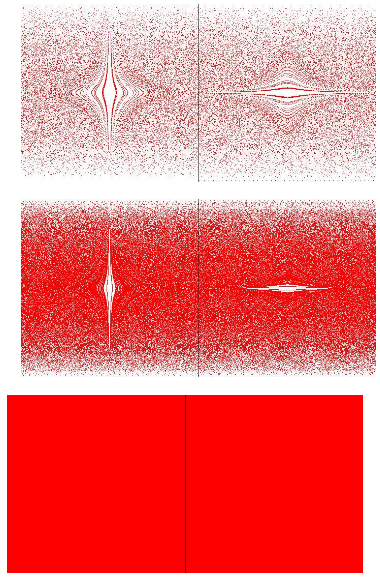

Figure 17: Trajectory segments on the phase space of the table $B = 1$, $R = 1.01$.

them are parabolic. As R increases, the table $Q(R, B)$ approaches the round disk table D_1 and many parabolic orbits of D_1 survive on the table $Q(R, B)$ (see Fig. 19). These orbits correspond exactly to those MUPOs in [2] and make the convergence even slower.

7.3.3 Tables with parameter in the interior of Region I

In this subsection we investigate ergodic property of systems with parameters in Region I. Note that from Fig. 10, for tables in this region the center of D_R lies outside the table $Q(R, B)$. For simplicity, we fix $R > 1$, and let B vary in the range $1 < B < R$. We will denote such a table by $Q(B)$ to indicate the dependence of the table on the parameter B, see Fig. 20.

To demonstrate the properties of the dynamics on such tables, let us examine the cases with $R = 1.5$ fixed, and let B vary from 1.01 to 1.49. We observe that, for most billiard tables in this process (except a short interval [1.2875,1.3025] of B's), the whole phase space is filled in by the trajectory of a single point. Moreover the distributions of the iterations are indistinguishable for several different choices of initial points, as long as the number of total iterations is larger than $10,000$. This implies that the dynamics on all three billiard tables should be ergodic. Fig. 21 shows the case with $B = 1.125$ (after $1,000,000$ and $10,000,000$ iterations, respectively). If we change the value of B to $B = 1, 1.25, 1.375$, the phase spaces of the tables $Q(B)$ behaves similarly with $Q(1.125)$.

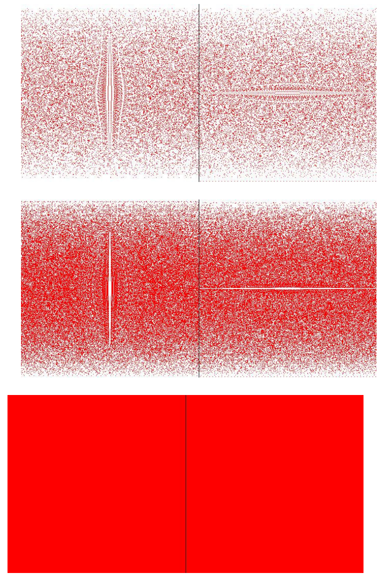

Figure 18: Trajectory segments on the phase space of the table $B = 1$, $R = 1.11$.

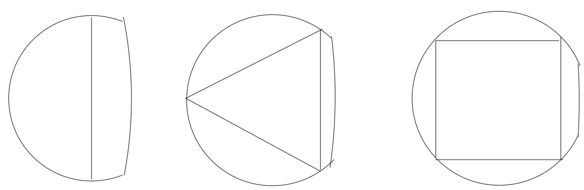

Figure 19: More and more parabolic periodic orbits appears when the table $Q(R, B)$ approaches the round disk table.

7.3.4 The exceptions on Region I

As mentioned in Section 7.3.2 and Section 7.3.3, we did identify a small set of parameters (R, B), among which the billiard dynamics on the tables $Q(R, B)$ exhibit elliptic islands while satisfying $R > B \geq 1$ surrounding some elliptic periodic orbit (see Fig. 22). More precisely, denote by $\mathcal{O}_3 = \{x_i = (s_i, \varphi_i) : i = 0, 1, 2\}$ the periodic-3 trajectory on $Q(R, B)$ with x_0 and x_1 sitting on the same arc. One can calculate the trace of the derivative DT^3 along this orbit, which is given by

$$\begin{aligned} \mathrm{Tr}(D_{x_0}T^3) &= 2 + \frac{8}{dd_1}(\tau_1 - d - d_1)(\tau_1 - d/2) \\ &= -2 + \frac{8}{dd_1}(\tau_1 - d)(\tau_1 - d_1 - d/2), \end{aligned} \tag{7.2}$$

where $d = \cos\varphi_0$, $d_1 = R\cos\varphi_2$, $\tau_1 = \tau(x_1)$.

Proposition 7.3. *Let $\mathcal{O}_3$ be the periodic orbit given in Fig. 22. Then the orbit $\mathcal{O}_3$ is elliptic if*

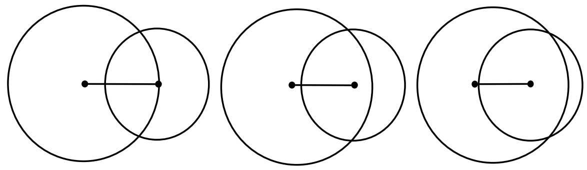

Figure 20: Decrease the distance between the centers.

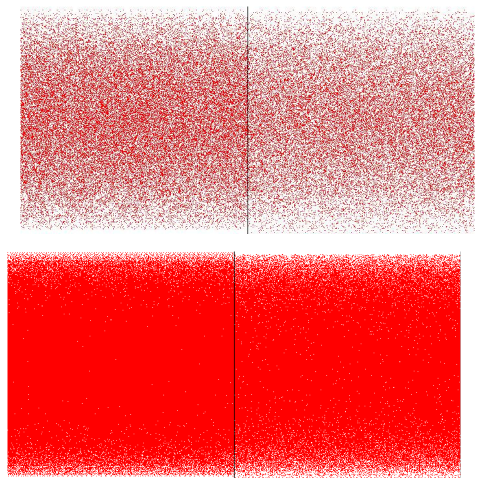

Figure 21: Trajectory segments on the phase spaces of the billiard table with $R = 1.5$ and $B = 1.125$.

and only if

$$\tau_1 > d_1 + d/2. \tag{7.3}$$

Proof. Note that $\tau(x) < d + d_1$ always holds on our table. Moreover it is easy to see $\tau_1 > d$ for this period-3 orbit (by drawing a perpendicular line from the center to τ_1). Combining terms, we see that $\mathrm{Tr}(D_{x_i}T^3) < 2$ by the first equality in Eq. (7.2). Thus the orbit $\mathcal{O}_3$ is elliptic if and only if $\mathrm{Tr}(D_{x_i}T^3) > -2$, which is equivalent to $\tau_1 - d_1 - d/2 > 0$ by the second equality in Eq. (7.2). $\square$

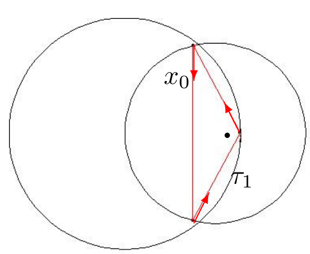

Figure 22: Period-3 orbit in the configuration space of the table $Q(R, B)$. It is elliptic if and only if $\tau_1 < d_1 + d/2$.

Note that the condition (7.3) fails on the majority of tables $Q(R, B)$ with $1 \leq B \leq R$, since $d_1 \sim R$ can be large. Therefore the periodic orbit $\mathcal{O}_3$ may not cause much problem for the ergodicity of most of billiard tables in Section 7.3.2 and Section 7.3.3. But there do exists a tiny region on which such a periodic orbit $\mathcal{O}_3$ exists and (7.3) holds along this orbit, which may fail the ergodicity of the billiard dynamics by Proposition 7.2. For example, if $R = 1.27$, then $\tau_1 > d_1 + d/2$ holds for all $B \in [1, 1.0198]$.

A generic feature about elliptic periodic orbit is that it is surrounded by infinitely many elliptic and hyperbolic periodic orbits of higher periods. See Fig. **??**, where we show two periodic orbits of higher periods near $\mathcal{O}_3$.

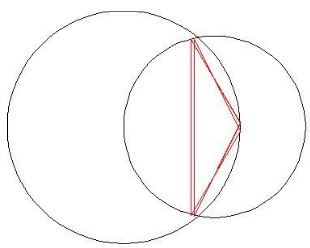

Figure 23: Doubling bifurcation: Period=6 on $Q(1.27, 1)$

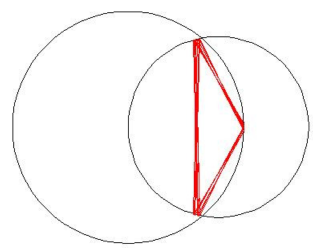

Figure 24: Further bifurcation: Period=24 on $Q(1.27, 1)$

Moreover, we do observe a small elliptic island around the elliptic orbit $\mathcal{O}_3$ in the phase space of the billiard table, see Fig. 25. Elliptic periodic orbits persist after small perturbations, so are the surrounding invariant curves with Diophantine rotation numbers. Therefore all tables in the nearby region of $(R, B) = (1.27, 1)$ should not be ergodic.

Remark 7.4. *We mainly focus on the periodic orbits of lower periods to get quantitative results. There might exist elliptic periodic orbits of higher periods. On the one hand, it is difficult to observe these orbits since the surrounding elliptic islands (if exists) might be too small and even invisible. On the other hand, periodic orbits of higher periods are sensitive to the initial conditions, may go through bifurcations (even cease to exist) after a very small changes.*

Similar phenomena are also observed on the tables with $(R, B) = (1.28, 1.02)$, $(1.3, 1.03)$, $(1.32, 1.06)$, $(1.36, 1.12)$ $(1.39, 1.16)$, $(1.40, 1.17)$ $(1.41, 1.19)$, $(1.43, 1.21)$, $(1.46, 1.25)$ and $(1.50, 1.29.5)$. It is interesting to note that all these tables are enclosed by two minor arcs. See Fig. 11 for the region of parameters in which the corresponding billiard systems admit this elliptic periodic orbit. A common feature of these tables is that the two components of their boundaries are minor arcs.

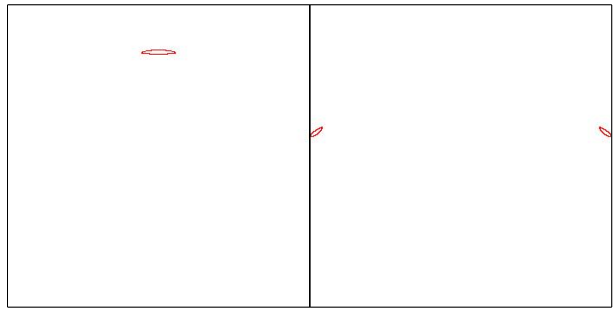

Figure 25: The phase space of $B = 1$ and $R = 1.27$, with elliptic islands surrounding the periodic orbit.

All the tables $Q(R, B)$ with one major-arc are ergodic by our simulation. This also provides an motivation for our ongoing project stated in the introduction.

7.4 Region II and III: Non-ergodicity and phase transitions

We have seen in the previous section that the dynamics on 'most' billiard tables $Q(R, B)$ in Region I are ergodic. In this section we will see that, the dynamics on every billiard table in Region Ⅱ and Region Ⅲ fails to be ergodic definitely. This completes the picture of different behaviors for the generalized lemon tables in Fig. 11.

First, it has been observed in the lemon billiard case [68, 73] (that is, $R = 1$), there exists an elliptic island surrounding the periodic orbit $\mathcal{O}_2$ for all $B \neq 1$). As we will see in the following subsections, this island undergoes significant developments as the parameter B moves away from 1.

7.4.1 Tables on the left boundary of Region II

We first let $B = R = 1$, and then increase B gradually. From Fig. 26 and 27 we can see that, for $1 < B \leq 1.35$, the phase space of the lemon billiard $Q(B)$ consists of exactly one chaotic component and one elliptic island centered at $\mathcal{O}_2$ (see Section 7.2). Moreover, the size of the island grows larger and larger as we keep increasing the distance B. New elliptic islands start to formulate when B goes over than 1.35. These new islands are centered around two periodic orbits or period six, see Fig. 28.

Orbits of the type appeared in Fig. 28 will persist for many larger values of B, but the shapes of the corresponding islands undergo some interesting transformations. One such example is visualized in Fig. 29, 30, 31, in which we provide three phase spaces with $B = 1.482$, 1.485 and 1.487. In these figures, new islands are created inside the the island centered at the periodic points with period 6 when we increase B. Moreover, as B continues growing, these new-formed islands get separated from the main island and form several isolated islands. This phenomenon corresponds to a *period-tripling effect*, that is, these new islands are related to orbits of period 18. These periodic orbits are highly unstable and no longer exist when the parameter B reaches 1.5.

By increasing B from $B = 1.5$ to $B = 1.6$, a similar pattern, the birth and separation of new islands, is observed in the phase space of the billiard table $Q(B)$. New small islands first appear within the main center island. And they move outward as B grows larger, eventually get separated from the main center island. See Fig. 32, 33 and 34.

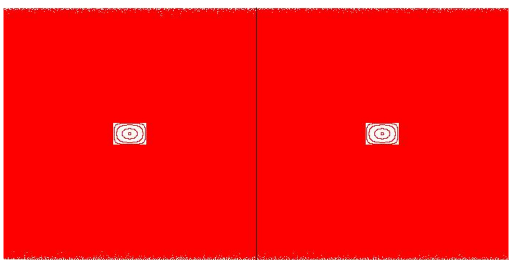

Figure 26: $B = 1.01$

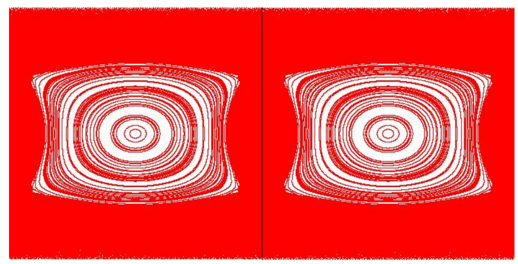

Figure 27: $B = 1.35$

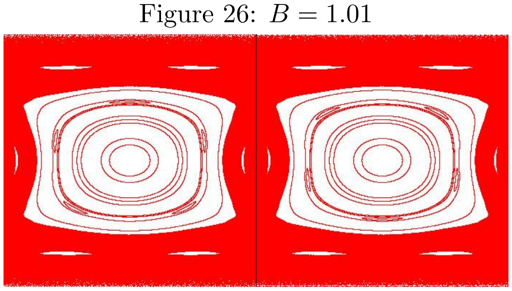

Figure 28: $B = 1.37$

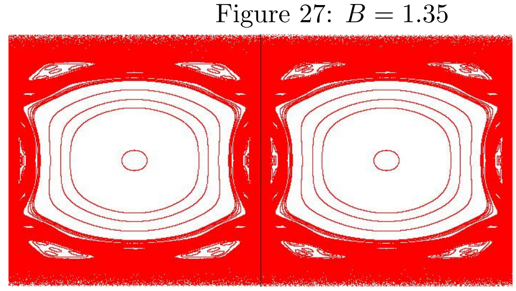

Figure 29: $B = 1.482$

As the parameter B continues to increase, the top and bottom paths of these orbits approach the corners of the table, eventually leading to their geometric destruction and the disappearance of the related islands in phase space. Finally, as B approaches 2 the ergodic portions of the phase space shrink significantly (see Fig. 35). We get a degenerate table when $B = 2$ and an empty table when $B > 2$ since the two circles no longer intersect.

These islands correspond to period orbits. For example, a pair of periodic six orbits illustrated in Fig. 36 and 37.

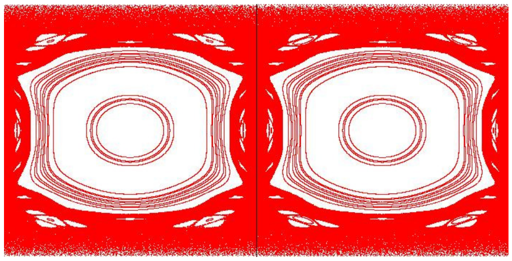

Figure 30: $B = 1.485$

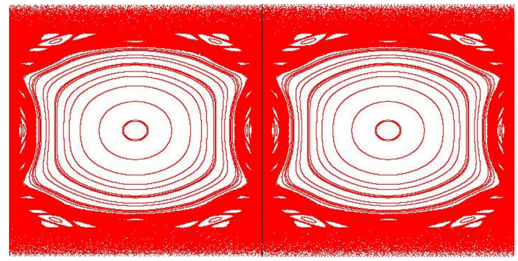

Figure 31: $B = 1.487$

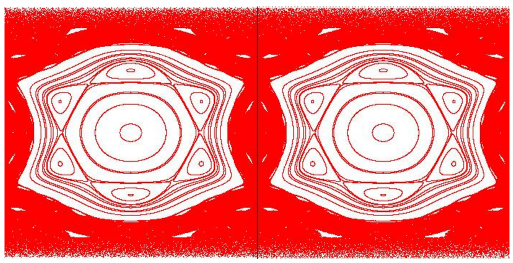

Figure 32: $B = 1.54$

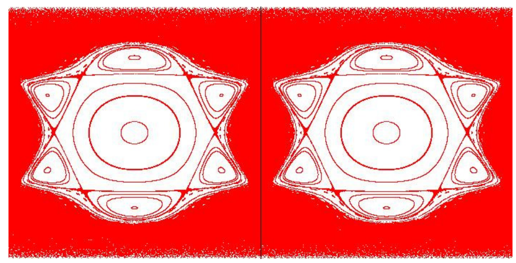

Figure 33: $B = 1.56$

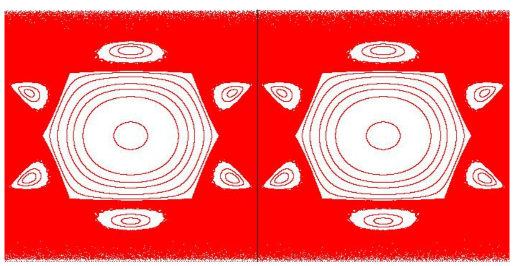

Figure 34: $B = 1.58$

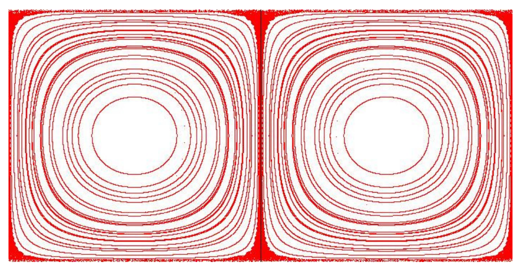

Figure 35: $B = 1.999$

7.4.2 Tables on the left boundary of Region III

Now let's move to the tables $Q(R, B)$ with $R = 1$, B varies from 1 to 0. We can see the bifurcation phenomena from a completely chaotic table ($B = 1$) to a completely integrable table (the unit disk, $B = 0$). We will see that as B approaches zero, more and more polygonal periodic orbits appear in our tables. These orbits strongly resemble those which appear in the circle billiard table.

As in the large distance case, the period two orbit of the type seen in Fig. 12 creates islands in phase space which will persist for all choices of B we consider here (see Fig. 38 and 39).

Moreover, for all tables with $R = 1$ and $B < \sqrt{2}$, there are two special periodic orbits (the

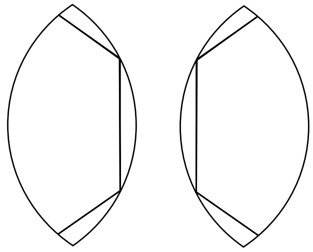

Figure 36: The orbits corresponding to the outlying islands in Fig. 28 with $R = 1$, $B = 1.37$.

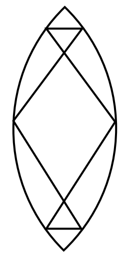

Figure 37: Period 6 orbit corresponding to Fig. 34 with $R = 1$, $B = 1.58$.

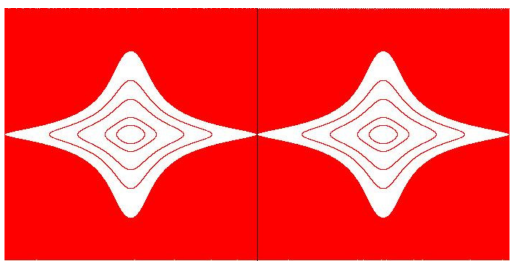

Figure 38: $B = 0.99$

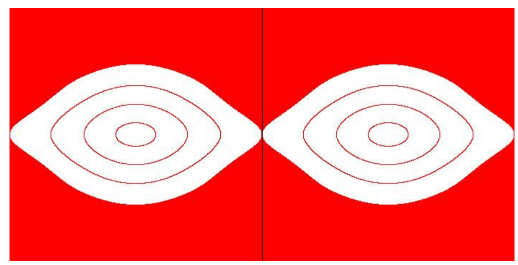

Figure 39: $B = 0.75$

square orbits of period 4 in Fig. 46), one given by

$$(s_1, \frac{\pi}{4}) \to (-s_1, \frac{\pi}{4}) \to (s_2, \frac{\pi}{4}) \to (-s_2, \frac{\pi}{4}) \to (s_1, \frac{\pi}{4}),$$

and another one given by reversing the direction of the trajectory. These periodic orbits disappear as B goes above the critical value $\sqrt{2}$ due to the geometric destruction.

Proposition 7.5. *This 4-period orbit of the billiard map on the table $Q(1, B)$ is hyperbolic if $B > 1/\sqrt{2}$, is parabolic if $B = 1/\sqrt{2}$ and is elliptic if $B < 1/\sqrt{2}$.*

Proof. Firstly we note that the travel time from $(s_1, \frac{\pi}{4})$ to $(-s_1, \frac{\pi}{4})$ is $\tau_0 = \sqrt{2}$, and the time from $(-s_1, \frac{\pi}{4})$ to $(s_2, \frac{\pi}{4})$ is $\tau_1 = \sqrt{2} - B$. As usual we compute the derivative matrix of T^4 at this periodic orbit and find the trace formula

$$\mathrm{Tr}(D_{(s_1, \frac{\pi}{4})} T^4) = \left(2 - \frac{4\sqrt{2}B}{1}\right)^2 - 2. \tag{7.4}$$

In particular $\mathrm{Tr}(D_{(s_1, \frac{\pi}{4})} T^4) > 2$ if $B > \frac{1}{\sqrt{2}}$, and the hyperbolicity of this periodic orbit follows. The other two conclusions follow from (7.4) similarly. $\square$

We can see from Proposition 7.5 that the hyperbolicity of these orbits get weaker and weaker as we decrease B (while keeping $R = 1$). Then these orbits lose their hyperbolicity and become parabolic orbits exactly at $B = 1/\sqrt{2}$. Finally they turn into (and stay as) elliptic orbits after d passes this critical value (until B reaches 0, at when the billiard table is a round table and all orbits are parabolic). We can also see this transformation from Fig. 40 to Fig. 43, that the periodic orbit $\mathcal{O}_4$ gets separated from the chaotic sea and develops an elliptic island around it (the four thin islands surrounding the main island in Fig. 42, whom develop to thick islands in Fig. 43).

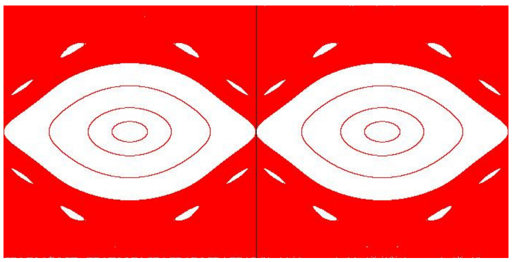

Figure 40: $B = 0.73$

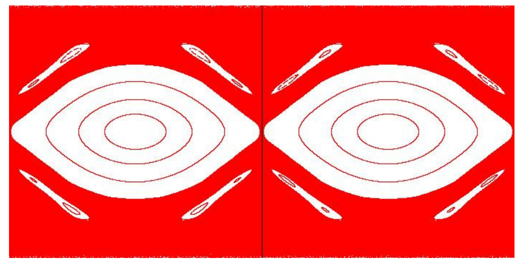

Figure 41: $B = 0.72$

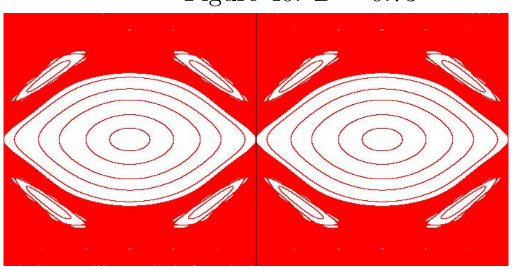

Figure 42: $B = 0.7$

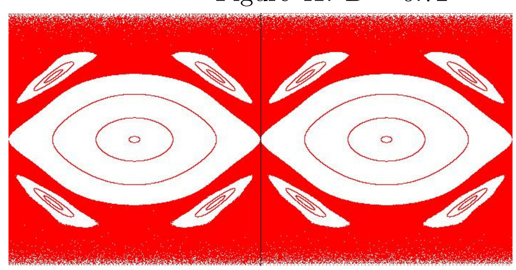

Figure 43: $B = 0.69$

As B decreases, the current islands keep developing and some new elliptic periodic orbits and the corresponding elliptic islands emerge. See Fig. 44, for $B = 0.5$, where we can observe the new island surrounding the periodic orbit given by Fig. 47; and Fig. 45 for $B = 0.3$, where new islands emerges surrounding the periodic orbits given by Fig. 48. Similar catalogue of periodic

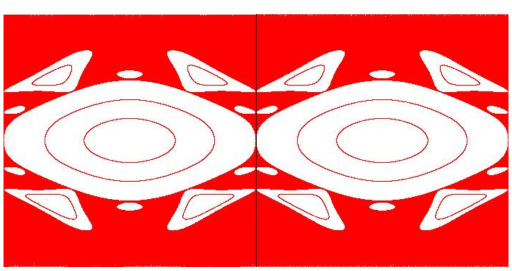

Figure 44: $B = 0.5$

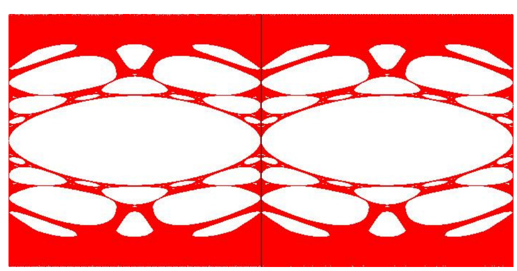

Figure 45: $B = 0.3$

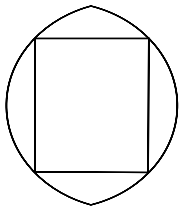

Figure 46: Square orbit for $R = 1$, $B < \sqrt{2}$.

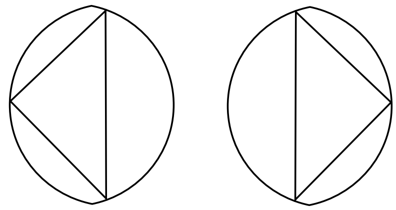

Figure 47: Triangular periodic orbits for $R = 1$, $B = 0.5$

orbits have also been observed on lemon-shaped billiards with parabolic boundary arcs [65] and with elliptical hyperbolic boundary arcs [66].

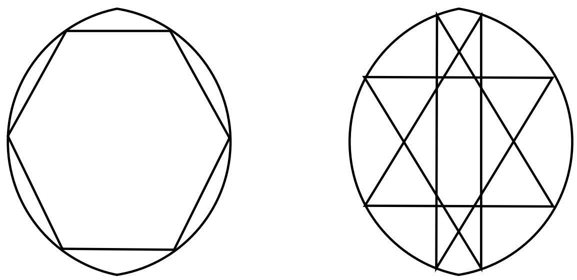

Figure 48: The hexagonal and period 8 orbits on the table with $R = 1$, $B = 0.3$

The table approaches the circular table as we continue decreasing B. This trend is clear from Fig. 49 and 50, where more islands appear in this process. In the phase space of the billiard table with $B = 0.01$, the islands are getting more "flattened", and approach horizontal lines as B shrinks. Finally for $B = 0$, each island has been completely flattened, that is, the phase space is foliated by horizontal lines. From the investigation of this class of billiards, it is clear that the periodic orbits in these tables and their corresponding islands in phase space play a crucial role in the transition from an ergodic billiard table to a table on which the dynamics is completely integrable.

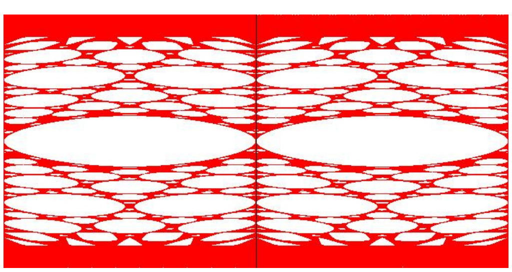

Figure 49: $B = 0.1$

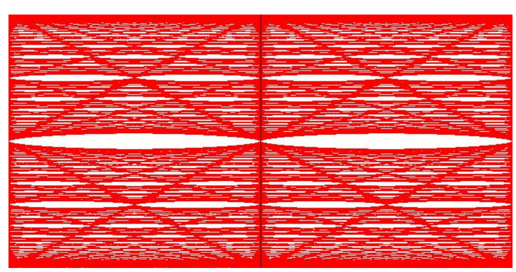

Figure 50: $B = 0.01$

7.4.3 Tables in the interior of II and III

We observe a similar dynamical behavior on the tables in the interior of Region II ($1 < R < B$) and Region III ($B < 1 < R$). Fig. 51 shows the trajectory segments of first 100,000,000 iterations on phase spaces for on the tables $Q(R, B)$ with $(R, B) = (1.0101, 1.0202)$, $(1.1111, 1.2222)$ and $(1.4286, 1.8571)$, respectively. Fig. 52 shows the trajectory segments of first 100,000,000 iterations on phase spaces for on the tables $Q(R, B)$ with $(R, B) = (1.1111, 0.8888)$, $(1.1111, 0.6667)$ and $(1.25, .05)$, respectively. These figures resemble those in [68, Fig. 4] and [73, Fig. 4], they just lose the symmetry of the distributions of the trajectory segments on the phase space when $R = r = 1$.

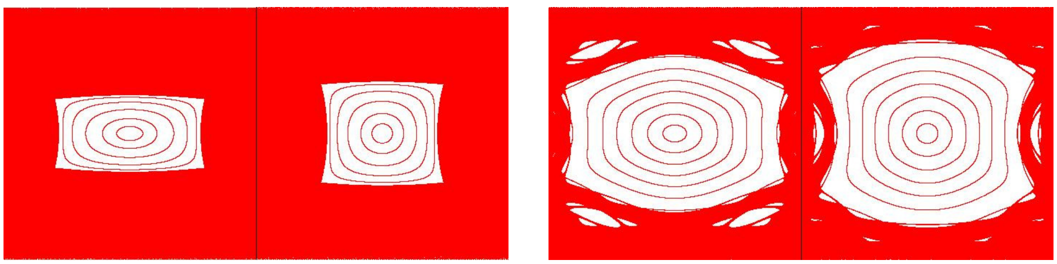

Figure 51: Trajectory segments on the phase space of the table $Q(R, B)$ with parameters $(R, B) = (1.1111, 1.2222)$ and $(1.4286, 1.8571)$, respectively.

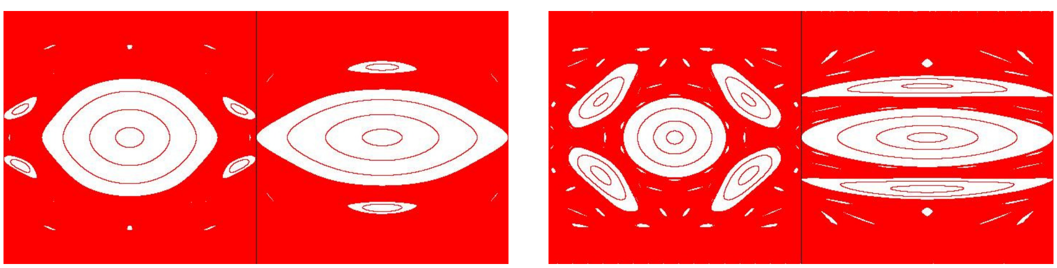

Figure 52: Trajectory segments on the phase space of the table $Q(R, B)$ with parameters $(R, B) = (1.1111, 0.6667)$ and $(1.25, .05)$, respectively.

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